On special elements for p-adic representations and higher rank Iwasawa theory at arbitrary weights

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On Special Elements for p-adic Representations and Higher Rank Iwasawa Theory at Arbitrary Weights

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A thesis presented for the degree of

Doctor of Philosophy in Pure Mathematics

June 2018

Abstract

In this thesis, we develop a theory of special elements in the higher exterior powers (or, more precisely, in the higher exterior power biduals) of the Galois cohomology of general p-adic representations over number fields.

These elements constitute a natural extension of the concept of a 'higher rank Euler system' and we present evidence that they encode detailed information about the structure of Galois cohomology groups.

In particular, we prove that a canonical ideal that one can define in terms of these elements is contained in both the relevant higher Fitting ideal and the annihilator ideal of the associated Galois cohomology group. In fact, under mild hypotheses, we find that the special elements completely determine the relevant higher Fitting ideal of the cohomology groups.

Building upon this result, we are then able to determine the complete structure of the torsion part of the quotient of the higher exterior powers of the Galois cohomology group modulo the subgroup generated by the special elements.

By means of a first concrete application, we specialise our theory to the p-adic representations that arise from the Tate motives with cyclotomic twists. In this way, we both recover and refine the theory of generalised Stark elements recently developed by Burns, Kurihara and Sano. At the same time, we are able to answer a question explicitly raised by both Washington and Lang regarding the Galois structure of global units modulo cyclotomic units in abelian fields, and also strongly refine a result of El Boukhari regarding the Galois structure of higher algebraic K-groups. In the same way, we can also formulate conjectures concerning p-adic L-series that have been formulated in other settings in earlier work of Castillo and Jones and of Solomon.

In the last part of the thesis, we study the Iwasawa theory of generalised Stark elements. Burns, Kurihara and Sano have conjectured that these elements are related by an explicit family of congruence relations and we now provide a new interpretation of these congruences. More concretely, we prove that the validity of these congruences implies that the Iwasawa theoretical zeta element, whose existence was predicted by the higher rank (abelian) Iwasawa Main Conjecture formulated by the same authors, has a natural interpolation property concerning the leading terms of Dirichlet L-series at arbitrary even integers. As an explicit example, we prove an unconditional result when the ground field is the field of rational numbers.

To Mama, Baba and my Sisters

Acknowledgements

I would like to express my deepest gratitude to Prof. David Burns for sharing his mathematical expertise generously with me and, more importantly, for demonstrating the right attitude to be a researcher/mathematician with his perseverance and precaution. His passion for mathematics and constructive criticism have been a constant inspiration to me during my doctoral studies. It will always be a pleasure to be his student.

I benefited hugely from numerous discussions with Dr. Takamichi Sano at the beginning of my PhD. I would like to thank him for explaining carefully and patiently a couple of his articles to me. I would also like to thank Dr. Lassina Dembélé, Dr. Mahesh Kakde, Prof. Masato Kurihara and Dr. David Solomon for their encouragement and interest in my work.

I would not have pursued mathematics in the first place without the encouragement of Dr. Koopa Koo Tak-Lun, who introduced the wonderland of numbers to me when I was 16. I owe him a great debt for his lively and beautiful lectures.

My doctoral studies would not have been as enjoyable without my academic siblings Asuka Kumon and Yasin Zähringer. They have tolerated my imperfect character but have always been perfect friends to me. I am particularly grateful to Asuka for never failing to make me laugh again in times when I am stressed or upset.

I would like to thank my flatmates Antonio Cauchi and Momchil Konstantinov for all the great (non-)mathematical times we share together at home. Besides them, I am exceedingly lucky to have met many great people in London, for example, my officemates Tomasz Kosmala, Umesh and Alfred Chong, my sportmates Yuanchang and Liu and my friends outside mathematics Bo-yi, Russell, Vincent, Anthony, Sabrina and Nicky. They have made me both mentally and physically stronger.

This thesis has been proofread (very) carefully by Alexandre Daoud, Vincent Cheung and Asuka. I am extremely grateful to them for pointing out several errors in the drafts of this thesis and their advice on how to improve the exposition.

I would like to express a special thanks to all the students I had while being a TA, for not only making the effort to come to my 9am tutorials, but also for the very encouraging and kind words they wrote in the evaluations. I thank Prof. Konstanze Rietsch and Prof. Simon Salamon for having faith in my teaching and offering many opportunities to me. My teaching philosophy and style were strongly influenced by Dr. Roger Tribe and Prof. José Luis Rodrigo during my undergraduate years at Warwick - they have demonstrated how to deliver effective and intellectual lectures in an exceptionally engaging manner.

I would like to thank Dr. Robin Chapman and Dr. Henri Johnston for a very enjoyable viva and for their valuable improvements and corrections to this thesis.

My doctoral studies were kindly funded by the Engineering and Physical Sciences Research Council [EP/L015234/1] via the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London. I am very glad to be the first cohort of this wonderful programme.

Last but not least, I would like to thank my beloved parents and sisters Caroline and Joanne whom I can always rely on, for allowing me freely to realise my dream, for offering their unconditional love, and for literally everything that has nurtured and cultivated me into who I am now. They are my key motivation to do the best and become a son/brother that they can be truly proud of.

Contents

Chapter 1

Introduction

1.1 Previous results and conjectures

1.1.1 From the class number formula to Stark's conjecture

One of the most beautiful discoveries in modern number theory is the interplay between algebraic and analytic invariants. The earliest such instance was discovered by Dirichlet in 1839 when he studied the Dedekind zeta-function $\zeta_k(s)$ of a number field k. Let us first recall that the (complex-valued) Dedekind zeta-function is defined as

$$\zeta_k(s) = \prod_{\mathfrak{p}} \left(\frac{1}{1 - N_{k/\mathbb{Q}}(\mathfrak{p})^{-s}} \right)$$

for $\mathrm{Re}(s) > 1$, where the product is taken over all non-zero prime ideals $\mathfrak{p}$ of k. By a complex analytic argument, it was shown that $\zeta_k(s)$ admits a meromorphic continuation and has a simple pole at $s = 1$. The celebrated 'class number formula' of Dirichlet offered an explicit formula for the residue of $\zeta_k(s)$ at $s = 1$ as follows:

$$\lim_{s\to 1}(s-1)\zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2}h_kR_k}{\omega_k\sqrt{|D_k|}},$$

where r_1 and r_2 are the number of real and complex places of k respectively, h_k, R_k, ω_k and D_k are the class number, regulator, number of roots of unity and the discriminant of k.

The above formula is striking for two reasons. Firstly, it is an equality that links an analytically defined function with the purely algebraic invariants associated with a number field. Secondly, despite the fact that the Dedekind zeta-function is defined 'locally' in terms of the prime ideals of k, it turns out to also encode 'global' and deep information about the underlying number field.

Using the functional equation satisfied by $\zeta_k(s)$, the class number formula can be transformed to give the leading term $\zeta_k^*(0)$ of the Taylor expansion of $\zeta_k(s)$ at $s = 0$:

$$\zeta_k^*(0) = -\frac{h_kR_k}{\omega_k}. \tag{1.1}$$

In particular, the (presumed) transcendence of $\zeta_k^*(0)$ is given by the regulator of k, which is defined as the determinant of a square matrix whose entries are given by the logarithm of archimedean valuations of certain units of k.

In 1970s, Stark published a series of papers ([67], [68], [69], [70]) in which he proposed to generalise the formula (1.1) when the Dedekind zeta-function is replaced by an Artin L-function. To be more specific, if F/k is a finite Galois extension of number fields and ρ is an irreducible, finite dimensional, complex representation of $\mathrm{Gal}(F/k)$, Stark conjectured that the leading term of an Artin L-function $L_{F/k}(\rho, s)$ at $s = 0$ can be described in terms of the determinant of certain matrices whose entries are given by the logarithm of archimedean valuations of certain units that belong to F (which we will refer to as the 'Stark units' in this introduction). We note that if $\rho = \mathbf{1}$ is the trivial representation of $\mathrm{Gal}(F/k)$, then the associated Artin L-function $L_{F/k}(\mathbf{1}, s)$ is just the Dedekind zeta-function for k. In [70], Stark provided evidence to his conjecture in the cases when F/k is abelian and k is either $\mathbb{Q}$ or a quadratic imaginary field by making use of the existing theory of cyclotomic units and elliptic units.

The recent development of modern number theory and arithmetic geometry is strongly influenced by the Stark conjecture in (at least) the following two ways. Firstly, Stark himself in [70, Conj. 1] formulated a conjecture regarding the first derivative of an Artin L-function at $s = 0$ only. One may wonder whether this conjectural framework can be extended to higher derivatives at $s = 0$ (or the 'higher rank' cases in current terminology). Secondly, questions regarding the leading term of an L-function lie at the heart of the study of arithmetic geometry. For instance, the Birch and Swinnerton-Dyer conjecture, which is listed as one of the Millennium Prize Problems by the Clay Institute, seeks an analogous formula to (1.1) for the leading term of the L-function associated with an elliptic curve over $\mathbb{Q}$ at $s = 1$. It is also worth mentioning that the vast generalisation of all these existing (some conjectural) leading term formulas was proposed by the seminal work of Bloch and Kato in [3], which is now known as the 'Tamagawa Number Conjecture'. While very little is known to be true in this direction (especially in the cases when the order of vanishing of the relevant L-function is strictly greater than one), it is believed that a better understanding of the Stark conjecture (and its refinement on the higher rank cases) may shed light on these, perhaps the deepest, open questions in number theory.

1.1.2 The refined Stark conjectures of Rubin and of Burns, Kurihara and Sano

The first attempt to formulate a Stark-type conjecture for abelian, S-truncated L-functions $L_{F/k,S}(\rho, s)$ with higher order zeroes was achieved by Rubin in [63]. Although there were previous conjectures suggesting that Stark units would live in the higher exterior powers of the groups of global units, Rubin observed and presented a counter-example in [63, §4] that it was too optimistic to expect that Stark units possesed no denominators at all. Nevertheless, Rubin in [63, Conj. B'] suggested that one could have a good control of the denominators of Stark units by proposing a lattice that (in most cases properly) contains the higher exterior powers of the global units. This led to what is usually referred as the 'Rubin-Stark conjecture' nowadays and the Stark units involved in this (possibly) higher rank scenario are often referred as the 'Rubin-Stark elements'. We will review the precise statement of the Rubin-Stark conjecture in §4.1.

While Rubin has proven some very special cases of his refined conjecture (mostly by building upon the work of Stark), very little was known about this conjecture until the important work of D. Burns in [8]. In loc. cit., Burns has shown that the Rubin-Stark conjecture is a consequence of the very general Equivariant Tamagawa Number Conjecture (eTNC). The latter conjecture was an equivariant refinement of the aforementioned conjecture of Bloch and Kato and concerned the leading term of an 'equivariant' L-function. In the case when the relevant Galois group is abelian, it was formulated by Kato in [44], [46] and independently by Fontaine and Perrin-Riou in [35]. In the case when the Galois group is not necessarily abelian, the eTNC was formulated by Burns and Flach in [13]. We remark that although the eTNC is still highly conjectural, it has been used extensively to make very precise predictions on the structures of arithmetic objects in terms of some analytic invariants, and many of these predictions have been verified either theoretically or numerically in many interesting cases. For instance, thanks to the work of Burns in [8] and his previous joint work with Greither in [27] and Flach in [33] on the eTNC for Tate motives, the Rubin-Stark conjecture is known to be valid unconditionally whenever F is abelian over $\mathbb{Q}$.

In [16], Burns, Kurihara and Sano discovered that the Rubin-Stark elements might encode detailed structural information of certain natural integral Selmer groups for $\mathbb{G}_m$ as Galois modules. In particular, they formulated a conjecture [16, Conj. 7.3] that the Rubin-Stark elements would generate the higher Fitting ideals of a relevant Selmer group and proved that the conjecture is implied by a relevant case of the eTNC. Not only does their conjecture refine the integral conjecture proposed by Rubin, but it also highlights the significance of Rubin-Stark elements in the theory of Galois modules. Along with all this, the authors also constructed the 'Weil-étale cohomology complex for $\mathbb{G}_m$' and reformulated the eTNC for untwisted Tate motives in a relatively concrete form. Their work will be surveyed in §4.3.

In this thesis, we intend to develop a theory of 'higher special elements' that arise from the category of 'strictly admissible complexes' in the higher exterior powers of the Galois cohomology of general p-adic representations and we present evidence that they encode detailed information about the structure of the relevant Galois cohomology groups. In the rest of this introduction, we will present several applications of our theory that would either resolve some classical problems or refine existing results in the current study of the (abelian) Stark conjectures.

1.2 Stickelberger-type annihilators for ideal class groups

Another direction in which to refine the Stark conjecture is to combine it with the ideas from a classical result of Stickelberger. First we review the statement of Stickelberger's Theorem. Let F/k be a finite abelian extension of number fields with Galois group G and let S be a finite set of places of k that contains all the archimedean and all those that ramify in F.

Write μ_F and $\mathrm{Cl}_S(F)$ for the subgroup of roots of unity in $F^\times$ and the S-class group of F respectively. If we further write $\theta_{F/k,S}(s)$ for the S-truncated equivariant L-function for F/k, then the theorem of Stickelberger says the following.

Theorem 1.2.1 (Stickelberger)**.** If F is abelian over $\mathbb{Q}$, then one has

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_F) \cdot \theta_{F/k,S}(0) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_S(F)).$$

However, the above theorem gives rise to non-trivial annihilators of S-class groups only in very limited cases because it is easy for $\theta_{F/k,S}(0)$ to vanish. Therefore, one may wonder if we can construct annihilators of the class groups using the derivatives of $\theta_{F/k,S}(s)$. Questions in this direction were previously considered by, amongst others, Rubin [62], Macias Castillo [51], Burns and Sano [22].

To present our main theorem in this direction, we first recall that $X_{F,S}$ is defined to be the kernel of the natural augmentation map $\bigoplus_{v\in S_F}\mathbb{Z} \to \mathbb{Z}$ (here S_F is the set of places of F above those in S) and that the (S,T)-unit $\mathcal{O}^\times_{F,S,T}$ of F (its precise definition is given in the beginning of §2.4.1) is a torsion-free subgroup of finite index of the S-units $\mathcal{O}^\times_{F,S}$. Write $\mathrm{Cl}^T_S(F)$ for the ray class group of S-integers of F modulo places in T.

Theorem 1.2.2. [Theorem 5.1.1] Let $\eta_{\mathcal{X}}$ be the 'higher special element' constructed from the 'Weil-étale cohomology complex' for F/k and any subset $\mathcal{X}$ of $X_{F,S}$. Set $a := |\mathcal{X}|$. Then under mild hypotheses on the choice of a subset S' of S and x in $\mathbb{Z}[G]$, for any Φ in $\bigwedge^a_{\mathbb{Z}[G]} \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])$, one has $\Phi(x\cdot\eta_{\mathcal{X}}) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S'}(F))$.

The precise statement of the above theorem will be made and proven in Theorem 5.1.1. In the situation when the relevant case of the eTNC is valid and a specific subset $\mathcal{X}$ of $X_{F,S}$ is chosen, the 'higher special element' coincides with the Rubin-Stark element (see Proposition 4.1.7). In these cases, Theorem 1.2.2 provides a construction of annihilators of the (S',T)-class groups by using the Rubin-Stark elements.

1.3 Kummer's class number formula for cyclotomic fields

In the 19th century, Ernst Kummer announced his celebrated proof of Fermat's Last Theorem for regular primes (recall that a rational prime p is regular if it does not divide the class number of the p-th cyclotomic field). This led to his further investigation into the class numbers of cyclotomic fields. One profound discovery of Kummer was that the class number of the maximal real subfield of a cyclotomic field appeared as the index of a certain distinguished subgroup of 'cyclotomic units' inside the full group of global units (for a concrete statement, see [78, Th. 8.2]).

A natural question that arises would be whether the class group of such fields is isomorphic to the quotient formed by the group of global units modulo the subgroup of cyclotomic units as Galois modules. Unfortunately this was false and Washington had offered a counter-example in [78, Rem. p.146]. In [49, p.260], Lang also commented that despite the well-known structures of both the groups of global units and cyclotomic units as Galois modules, the module structure of their quotient was a 'mystery'. One breakthrough in this direction was achieved by Cornacchia and Greither in [27]. They had shown that if p is an odd prime, $F = \mathbb{Q}(\mu_{p^n})^+$ and $G = \mathrm{Gal}(F/\mathbb{Q})$, then there is an equality of Fitting ideals

$$\mathrm{Fit}^0_{\mathbb{Z}[G]}(\mathrm{Cl}(F)) = \mathrm{Fit}^0_{\mathbb{Z}[G]}(\mathcal{O}_F^\times/\mathbb{Z}[G]\cdot c_F)$$

where c_F is a certain cyclotomic unit of F. In this thesis, we will prove the following theorem (here we use the T-modified global units and class groups as introduced in §1.2).

Theorem 1.3.1. In the situation described above, there is an isomorphism of (cyclic) $\mathbb{Z}[G]$-

modules

$$(\mathcal{O}_{F,S,T}^{\times}/\langle c_{F,T}\rangle)^{\vee} \cong \mathbb{Z}[G]/\operatorname{Fit}^0_{\mathbb{Z}[G]}(\mathrm{Cl}_S^T(F)),$$

where $(-)^{\vee}$ denotes the Pontryagin dual endowed with the natural (rather than contragradient) action and $c_{F,T}$ is a certain 'T-modified' cyclotomic unit of F.

A more general form of the above theorem will be proven as Theorem 5.2.1. We also remark that this isomorphism allows us to construct a perfect G-invariant pairing of the form

$$(\mathcal{O}_{F,S,T}^{\times}/\langle c_{F,T}\rangle)_{\mathrm{tor}} \times (\mathbb{Z}[G]/\mathrm{Fit}^1_G(\mathrm{Sel}_S^T(F)))_{\mathrm{tor}} \to \mathbb{Q}/\mathbb{Z},$$

where $\mathrm{Sel}_S^T(F)$ is the aforementioned integral Selmer module for $\mathbb{G}_m$. See Remark 5.2.2 for details of this construction.

1.4 Other weights and higher K-groups

Another direction in which to generalise the formula (1.1) is to study the leading term of the zeta functions at other integers (or other 'weights' in motivic language). One attempt was to use algebraic K-theory developed in the 1950s. Recall that for any Dedekind domain R, there are natural isomorphisms of the K-groups $K_0(R) \cong \mathbb{Z} \oplus \mathrm{Pic}(R)$ and $K_1(R) \cong R^{\times}$. Inspired by this and Borel's definition of 'higher regulators' Reg_n^B, Lichtenbaum then conjectured that for any number field k and $n > 0$, the ratio $\zeta_k^*(1-n)/\mathrm{Reg}_n^B$ coincides with $|K_{2n-2}(\mathcal{O}_k)|/|K_{2n-1}(\mathcal{O}_k)_{\mathrm{tor}}|$ up to a sign and a power of 2.

Therefore, it is natural for us to seek an analogous statement of Theorem 1.3.1 with the group of global units (resp. the class group) replaced by an appropriate odd (resp. even) K-group. For brevity, we will discuss only a special case of our theorem in this direction. To do this, we set $\mathbb{Z}' = \mathbb{Z}[1/2]$ and for each abelian group write M' in place of $\mathbb{Z}' \otimes_{\mathbb{Z}} M$.

We fix an integer f with $f \not\equiv 2 \pmod 4$, write $F = \mathbb{Q}(\mu_f)^+$ and $G = \mathrm{Gal}(F/\mathbb{Q})$. Let Σ be a finite set of places of $\mathbb{Q}$ containing the unique infinite place and all places dividing f. We also write $\epsilon_m(\zeta_f)$ for Beilinson's 'cyclotomic element' in $\mathbb{Q} \otimes_{\mathbb{Z}} K_{2m-1}(\mathcal{O}_F)$, as described by Neukirch in [56, Part II, §1], and then set $c_F(m) := 2^{-1}(m-1)! f^{m-1} \cdot \epsilon_m(\zeta_f)$.

Theorem 1.4.1. For every odd integer $m > 1$, there exists an isomorphism of $\mathbb{Z}'[G]$-modules

$$\left(\frac{K_{2m-1}(\mathcal{O}_F)'}{\mathbb{Z}'[G] \cdot c_F(m)}\right)^{\vee} \cong \frac{\mathbb{Z}'[G]}{\mathrm{Fit}^0_{\mathbb{Z}'[G]}(K_{2m-2}(\mathcal{O}_{F,\Sigma})')}.$$

A more general version (which deals with any integer $m > 1$) of the above theorem will be formulated and proven as Theorem 5.3.3. In addition, we will explain the relation of the above theorem with the work of El Boukhari [31] regarding Fitting ideals of higher K-groups. Along with the proof, in §5.3.1, we will also make important connections of our theory with the recent theory of 'Stark elements of arbitrary ranks and weights' developed by Burns, Kurihara and Sano in [18].

1.5 A new conjecture for the values of p-adic L-series

Due to the functional equation satisfied by the (complex) Artin L-series, any arithmetic properties satisfied by the leading term at $s = 0$ can be easily reformulated in terms of an appropriate derivative at $s = 1$. However, no analogous equations are known to be satisfied by their p-adic counter-parts, for example, the p-adic L-series constructed by Kubota-Leopoldt. Therefore, it is of much interest to derive arithmetic properties of values (or even derivatives) of p-adic L-series at $s = 1$. The integrality and some Stickelberger-type (analogous to Theorem 1.2.1) annihilation results satisfied by the p-adic L-series at $s = 1$ were previously studied, amongst others, by Oriat [58], by Solomon [71], [72], and by Barrett and Burns [2]. In this thesis, we will derive from our theory of higher special elements a new conjecture regarding

p-adic L-series.

To formulate the conjecture, we introduce a couple of notations. Fix an odd prime p. For any Galois extension F/k of group G, we write $\mathbf{1}$ to be the trivial (p-adic) character of G and write $e_{\mathbf{1}}$ to be the associated idempotent. We set $A_F := \mathrm{Cl}(F) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ to be the p-part of the ideal class group. In the case when F/k is an abelian extension of totally real fields, we denote by $\mathcal{L}_{F/k,S,p}(1)$ the equivariant Kubota-Leopoldt p-adic L-series at the value $s = 1$.

Conjecture 1.5.1. Let F/k be an abelian extension of totally real number fields of Galois group G. Then the element $\mathcal{L}_{F/k,S,p}(1)$ belongs to $\mathbb{Z}_p[G](1 - e_{\mathbf{1}})$ and annihilates the module $A_F/H^0(G, A_F)$.

The statement of the above conjecture will be made precise in §5.4.2. To supply evidence of the above conjecture, we will prove in Theorem 5.4.4 and Remark 5.4.5 the following result.

Theorem 1.5.2. If F is a totally real abelian extension of $\mathbb{Q}$, then Conjecture 1.5.1 is valid.

We remark that the proof of the above theorem makes use of the known validity of the eTNC for the pair $(h^0(\mathrm{Spec}(F))(1), \mathbb{Z}_p[G])$ by Burns and Greither [15], of Leopoldt's Conjecture for F and of the 'p-adic Stark conjecture at $s = 1$' for F/k (as formulated in [25, Rem. 7.2]). The last mentioned conjecture, according to Burns and Venjakob in [25], plays an important role in the 'descent mechanism' of equivariant Iwasawa theory to deduce important cases of the eTNC.

1.6 Congruences and Iwasawa theory

Besides the class number formula for cyclotomic fields, Kummer also derived by (1.1) a criterion for the regularity of a prime p in terms of the p-adic valuations of the values of the Riemann zeta function $\zeta(s)$ at negative integers. For instance, it is now known that 691 is an irregular prime because 691 divides the numerator of $\zeta(-11)$, even though it is rather unwieldy to compute the class group of $\mathbb{Q}(\mu_{691})$ explicitly. Along with these discoveries, Kummer also proved his celebrated congruences satisfied by the values of Riemann zeta function at negative integers. This aroused the curiosity of many forthcoming mathematicians who began to investigate the p-adic properties of special values of zeta-functions. For example, the Kummer congruences have played a vital role in the construction of the aforementioned p-adic L-series of Kubota-Leopoldt.

While Kummer's results concern only the p-adic orders of class numbers or zeta values, Iwasawa had a more ambitious goal to describe the Galois structure of the class group in terms of the relevant zeta values. The insight of Iwasawa was not to study the structure of a particular class group, but the 'limit' of all the class groups over a 'cyclotomic tower'. This ultimately led to his celebrated 'main conjecture' which, roughly speaking, says that there is a 'characteristic element' associated with the limit of class groups and this element interpolates special values of Dirichlet L-functions in the same way as the Kubota-Leopoldt's p-adic L-series. While the initial 'main conjecture' of Iwasawa had already been proven by, firstly Mazur-Wiles in [53] by modular methods, and later by Rubin, who built upon the idea of Kolyvagin, by using the 'Euler system' of cyclotomic units, the philosophy of Iwasawa has been adopted and generalised to study the Galois structure of many other p-adic representations and remains the most powerful tool in attacking problems in special values of various L-functions.

In [17], inspired by the formulations of the 'Equivariant Tamagawa Number Conjecture' of Burns and Flach [13] and that of the 'generalised main conjecture' of Fukaya and Kato [36], Burns, Kurihara and Sano formulate a natural 'higher rank' main conjecture of Iwasawa theory over arbitrary (finite) abelian extensions of number fields. Their conjecture predicts the existence of an Iwasawa-theoretical zeta element that interpolates, in a natural way, the derivatives at zero of L-functions of Dirichlet characters and extends the classical formulation of main conjectures over totally real fields (as studied, for example, by Wiles in [80]). The formulation of their conjecture will be reviewed in §6.1.

In a subsequent article [18], the same authors associated to finite abelian extensions of number fields a natural notion of generalised Stark elements of arbitrary rank and weight. Whilst these elements belong, a priori, to the complexified higher exterior power of an appropriate cohomology group, the 'generalised Rubin-Stark Conjecture' formulated in [18, Conj. 3.5] predicts that they are, in a natural sense, both rational and have denominators that are bounded in a way that naturally extends the Rubin-Stark conjecture.

Assuming the bound on denominators that is predicted by the generalised Rubin-Stark Conjecture, the authors then formulate in [18, Conj. 3.12] a precise congruence relation between Stark elements of fixed rank and differing weights. They show that this explicit congruence conjecture specialises to recover a wide variety of results and conjectures in the literature including, amongst other things, the classical congruences of Kummer and the explicit reciprocity law of Artin-Hasse and Iwasawa [38]. This congruence conjecture will be reviewed in §4.4.

In the last part of this thesis (§6.2), we shall show that this congruence conjecture also has important consequences for the higher rank Iwasawa theory that was formulated in [17]. Specifically, in Theorem 6.2.1, we show that if all relevant cases of these congruences are valid, then the 'zeta element' that is only predicted by the central conjecture of [17] to have precise interpolation properties involving the leading terms at zero of Dirichlet L-functions will in fact also have analogous interpolation properties that involve the leading terms of Dirichlet L-functions at all even integers.

1.7 Future directions

In this section, we briefly discuss two directions for further research that are suggested by the results obtained in this thesis.

1.7.1 Towards a proof of Conjecture 1.5.1

Our proof of Theorem 1.5.2 relies on the known validity of the relevant case of the Equivariant Tamagawa Number Conjecture.

Given the proof (by Burns and Greither) of the latter result, it would seem reasonable to expect that, modulo the standard conjecture on the vanishing of cyclotomic μ-invariants, one could in fact give a full proof of Conjecture 1.5.1 by combining Wiles's proof of the Iwasawa Main Conjecture for totally real fields [80] together with the sort of results on derivatives of p-adic L-series that are proved by Burns in [11].

We hope to consider this problem in a future article.

1.7.2 Non-abelian generalisations

In the preprint [22], Burns and Sano have recently described a natural extension of their previous work with Kurihara concerning arithmetic zeta elements [16] to a non-abelian setting.

In particular, to do this they introduced in [22, Part I] natural non-commutative generalisations of the classical notions of determinant functor and higher Fitting invariants that play a key role in this thesis and used these extended notions to formulate a 'non-abelian refined Rubin-Stark conjecture'.

These algebraic techniques have already been adopted by Burns and Livingstone Boomla in [20] to formulate natural higher order, non-abelian Stark conjectures for Artin L-series.

It seems reasonable to expect that the theory of higher special elements developed in this thesis would have a natural non-commutative counterpart by using the algebraic technique introduced in [22]. We would also hope that such a generalised theory might enable one to understand the higher Fitting invariants and the structure of the exterior bidual of a relevant cohomology group in a non-abelian setting.

It would also be interesting to find the connection of such a theory with, firstly, the work of Burns in [9] in which, under the validity of the 'Strong Stark Conjecture', explicit annihilators of class groups are constructed by using the values at $z = 0$ of higher-order derivatives of an (equivariant) Artin L-series, and also the subsequent work of Johnston and Nickel in [41].

Regarding the analogous results for other weights, Nickel in [55] constructed (conjectural) annihilators for higher K-groups by using the leading terms of a relevant Artin L-series at negative integers and it seems possible that these results could also be refined by this sort of approach.

1.8 Structure of this thesis

We now give a brief outline of this thesis.

In Chapter 2, we collect the necessary algebraic results that will be used throughout this thesis. In particular, we will recall the definition of the determinant functor of Grothendieck, Knudsen and Mumford in [48] in §2.1 and its basic properties. The notion of higher Fitting ideals and their properties will be given in §2.2 in which we will closely follow [57]. In §2.3, we will introduce the category of 'admissible complexes' and we will prove in §2.3.3 a natural construction of a 'strongly admissible complex' from each object of this category. Then in §2.4, we will give explicit examples of admissible complexes that arise naturally from arithmetic settings, namely the Weil-étale cohomology for $\mathbb{G}_m$ and the compactly supported étale cohomology. Their properties will be reviewed respectively in §2.4.1 and §2.4.2. Along with these, we will review the definitions of (Bloch-Kato) Selmer and Tate-Shafarevich groups for p-adic representations and prove in Proposition 2.4.10 a construction of an admissible complex from the compactly supported étale cohomology for which the associated (Bloch-Kato) Tate-Shafarevich group appears as a subquotient of one of its cohomology groups. In §2.5, we will define the two kinds of dual modules that we will mostly deal with in this thesis and prove some basic algebraic properties of them in Lemma 2.5.2. Then in §2.6.1 we will recall the definition of the exterior power bidual functor and we will prove in Lemma 2.6.3 a result concerning the behaviour of exterior biduals under an inverse limit.

In Chapter 3, we give in §3.1 the definition of a 'higher special element' constructed from a strictly admissible complex C with a choice of an (ordered) subset of the torsion-free part of $H^2(C)$. We will also define what it means for the latter choice of subset to be separable in §3.1.3. Our first main result in this chapter is Theorem 3.2.1 (i) in which we prove that a canonical ideal that one can define in terms of these special elements is contained in both a higher Fitting ideal and the annihilator ideal of $H^2(C)$. We remark that this result neither assumes the underlying order is Gorenstein nor that the choice of the ordered subset of $H^2(C)$ is separable. In the case when these conditions are valid, we find that the special elements completely determine the relevant higher Fitting ideal of $H^2(C)$ (see Theorem 3.2.1(ii) and Remark 3.2.3). Building upon this result, we will then prove Theorem 3.3.1 in which we are able to determine the complete structure of the torsion part of the quotient of the higher exterior powers of $H^1(C)$ modulo the subgroup generated by the special elements. The proofs of Theorem 3.2.1 (i), (ii) and Theorem 3.3.1 will be given in §3.2.2, §3.2.3 and §3.3.2 respectively.

In Chapter 4, we give a survey on the current development of Stark conjectures, following closely the series of articles of Burns, Kurihara and Sano [16], [18]. In §4.1, we recall the definition and (known) examples of Rubin-Stark elements and formulate the precise statement of the Rubin-Stark conjecture in Conjecture 4.1.4. We will also formulate the Equivariant Tamagawa Number Conjecture for untwisted Tate motives and its relation with the Rubin-

Stark element. Assuming the validity of the last conjecture, we show in Proposition 4.1.7 that the Rubin-Stark element coincides with the higher special element associated with the Weil-étale cohomology complex and a specific choice of a separable subset. In §4.2, we will survey the theory of generalised Stark elements of arbitrary rank and weights developed by Burns, Kurihara and Sano in [18]. Analogous to the Rubin-Stark conjecture, we will discuss the connection of generalised Stark elements with the Equivariant Tamagawa Number Conjecture for (twisted) Tate motives and prove in Proposition 4.3.3 that in this case the generalised Stark elements can be recognized as the higher special elements associated with the dual of the compactly supported étale cohomology (and a specific choice of a separable subset). In §4.4, we will review the statement of the 'Generalised Kummer Congruence Conjecture' [18, Conj. 3.12] of Burns, Kurihara and Sano between generalised Stark elements of different weights and the known cases of these congruences.

In Chapter 5, we specialise the theory of higher special elements to the p-adic representations that arise from Tate motives with cyclotomic twists. In this way we shall, amongst other things, firstly make Theorem 1.2.2 precise and extend the existing theory of refined Stark conjectures in §5.1. Then we prove Theorem 5.2.1 which is a more general version of Theorem 1.3.1 and address a problem explicitly raised by Washington in [78, Remark after Th. 8.2] and by Lang in [49, p. 260] in Remark 5.2.2. In §5.3 we refine the main conjecture for generalised Stark elements purposed by Burns, Kurihara and Sano in [18] (see Remark 5.3.2) and formulate the 'correct form' and extend the main result proved by El Boukhari in [31] (see Remark 5.3.5). Finally in §5.4, we make Conjecture 1.5.1 (and hence Theorem 1.5.2) precise and give some evidence to this conjecture.

In Chapter 6, we study the Iwasawa theory of generalised Stark elements. In particular, we now provide in Theorem 6.2 a new interpretation of the 'Generalised Kummer Congruences'. More concretely, we prove that the validity of these congruences implies that the Iwasawa theoretical zeta element, whose existence was predicted by the higher rank (abelian) Iwasawa Main Conjecture formulated by Burns, Kurihara and Sano in [17], has a natural interpolation property concerning the leading terms of Dirichlet L-series at arbitrary even integers. As an explicit example, we prove an unconditional result in Theorem 6.4.1 when the ground field is the field of rational numbers.

1.9 General notation

For the reader's convenience, we collect here the notation that will be used in the thesis.

For a noetherian ring R, we write $D(R)$ for the derived category of R-modules. For an abelian group N, we write N_{tor} for its torsion submodule and set $N_{\mathrm{tf}} := N/N_{\mathrm{tor}}$, which we regard as embedded in the associated space $\mathbb{Q} \otimes_{\mathbb{Z}} N$. Let E be a field containing R. If A is a R-module, we sometimes denote $E \otimes_R A$ by simply A_E or $E \cdot A$. For any complex $C = (C^i)_{i \in \mathbb{Z}}$ in $D(R)$ and an integer j, we define the 'shifted complex' $C(j)$ to be the complex such that the module C^{i+j} is placed at degree i. For any R-module M and integer a, we write $M[a]$ to be the complex such that M is placed at degree $-a$ and the zero module is placed at any other degree.

Fix an algebraic closure $\mathbb{Q}^c$ of $\mathbb{Q}$. For any non-negative integer m, we denote by μ_m the subgroup of all m-th roots of unity in $(\mathbb{Q}^c)^\times$. For a rational prime p, we denote the inverse limit $\varprojlim_n \mu_{p^n}$ by $\mathbb{Z}_p(1)$. For $j > 0$, we set $\mathbb{Z}_p(j) := \mathbb{Z}_p(1)^{\otimes j}$ and for $j < 0$, we set $\mathbb{Z}_p(j) := \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(j), \mathbb{Z}_p)$. For any $\mathbb{Z}_p$-module T and integer j, we denote $T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(j)$ by $T(j)$.

For any finite group Γ and any $\mathbb{Z}_p[\Gamma]$ (resp. $\mathbb{Z}[\Gamma]$)-module N we write $N^\vee$ for the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(N, \mathbb{Q}_p/\mathbb{Z}_p)$ (resp. $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$) which we endow with the usual contragredient action of $\mathbb{Z}_p[\Gamma]$ (resp. $\mathbb{Z}[\Gamma]$). To be more specific, the contragradient action is defined by setting $\sigma \cdot f(x) = f(\sigma^{-1} \cdot x)$ for any $\sigma \in \Gamma$ and $f(x) \in \mathrm{Hom}_{\mathbb{Z}_p}(N, \mathbb{Q}_p/\mathbb{Z}_p)$ (resp. $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$).

For a number field k, we denote the set of archimedean and p-adic places of k by $S_\infty(k)$ and $S_p(k)$ respectively. Sometimes we write these sets as S_∞ and S_p if it is clear from the context what the underlying field is. For any place v of k, we denote its completion of k at the place v by k_v. We denote the maximal totally real subfield of k by k^+. For an extension L/k, we write $S_{\mathrm{ram}}(L/k)$ for the set of places of k that ramify in L. For any subset of places S of k, we denote by S_L the set of places in L above those in S and $\mathcal{O}_{L,S}$ for the ring of S_L-integers in L. For any place w of L above v of k, we denote its residue field by $\kappa(w)$, set $Nw := |\kappa(w)|$ and identify the decomposition subgroup of w in $\mathrm{Gal}(L/k)$ with $\mathrm{Gal}(L_w/k_v)$ in the usual way.

For an abelian group G, we write $\widehat{G}$ for the set of irreducible complex (linear) characters of G of finite order. If G is finite, then for each χ in $\widehat{G}$ we define an idempotent of $\mathbb{C}[G]$ by setting $e_\chi := |G|^{-1} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}$.

Chapter 2

Preliminaries

2.1 Determinant modules and perfect complexes

The determinant functor of Grothendieck, Knudsen and Mumford in [48] has played an important role in the formulation of several leading term conjectures in the literature. In this section, we briefly recall its construction and collect the important properties that will be used throughout this thesis.

Let R be a commutative noetherian unital ring. Then R can be decomposed into a direct sum of rings $R = \prod_{i \in I} R_i$ for some finite set I such that each $\mathrm{Spec}(R_i)$ is connected (with respect to the Zariski topology). For any finitely generated projective R-module P (equivalently a locally free module of finite rank), define $\mathrm{rk}_P : \mathrm{Spec}(R) \to \mathbb{Z}$ by

$$\mathfrak{p} \mapsto \mathrm{rank}_{R_\mathfrak{p}}(P \otimes_R R_\mathfrak{p}).$$

We remark that this function is continuous if we equip $\mathrm{Spec}(R)$ (resp. $\mathbb{Z}$) with the Zariski (resp. discrete) topology and hence locally constant on each connected component $\mathrm{Spec}(R_i)$. Hence, we can define for each $i \in I$ an integer $r_i(P) := \mathrm{rk}_P(\mathrm{Spec}(R_i))$.

Now we recall that the category of graded invertible R-modules consists of objects of the form (M, f) where M is an invertible R-module (equivalently locally free of rank one) and $f : I \to \mathbb{Z}$ is a continuous function. In this category, every object (M, f) has a natural inverse by setting $(M, f)^{-1} := (\mathrm{Hom}_R(M, R), -f)$. Moreover, for any pair of objects (M, f) and (N, g), one can define their tensor product by setting $(M, f) \otimes (N, g) := (M \otimes_R N, f + g)$.

Definition 2.1.1. For any finitely generated projective R-module P, the determinant module of P is a graded invertible R-module defined by setting

$$\mathrm{Det}_R(P) := \left(\bigoplus_{i \in I} \bigwedge_{R_i}^{r_i(P)} (P \otimes_R R_i), \mathrm{rk}_P \right).$$

If $C^\bullet := (C^i)_{i \in \mathbb{Z}}$ is a bounded complex of finitely generated projective R-modules, we define the determinant module of $C^\bullet$ by setting

$$\mathrm{Det}_R(C^\bullet) := \bigotimes_{n \in \mathbb{Z}} \mathrm{Det}_R(C^n)^{(-1)^n}.$$

Lemma 2.1.2. Let P be a finitely generated projective R-module.

(i) There is a canonical evaluation pairing $\mathrm{Det}_R(P) \otimes \mathrm{Det}_R^{-1}(P) \xrightarrow{\sim} (R, 0)$ induced by setting

$$a \otimes \Phi \longmapsto \Phi(a).$$

(ii) If there is a ring homomorphism $R \to R'$, then there is a canonical isomorphism

$$\mathrm{Det}_R(P) \otimes_R R' \cong \mathrm{Det}_{R'}(P \otimes_R R').$$

(iii) For any short exact sequence $0 \to P_1 \to P_2 \to P_3 \to 0$ of finitely generated projective

R-modules, there is a canonical isomorphism

$$\mathrm{Det}_R(P_2) \xrightarrow{\sim} \mathrm{Det}_R(P_1) \otimes \mathrm{Det}_R(P_3). \tag{2.1}$$

Proof. All the above properties can be found in [48]. We briefly sketch the argument here. (i) follows because the 'inverse determinant' $\mathrm{Det}_R^{-1}(P)$ is defined as the linear dual $\mathrm{Hom}_R(P, R)$. (ii) follows directly from the base change property of exterior powers. For (iii), it suffices to prove the statement when R is replaced by its connected component R_i. We can further replace R_i by its localisation $R_{i,\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of R_i. Recall that over a local ring $R_{i,\mathfrak{p}}$, a module is projective if and only if it is free. We label $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ to be bases of P_1 and P_3 respectively. We let $\{u_i\}_{i\in I}$ and $\{u'_j\}_{j\in J}$ denote the image and pre-image respectively of $\{v_i\}_{i\in I}$ and $\{w_i\}_{i\in J}$ under the given short exact sequence. The claimed isomorphism is then given by

$$(\wedge_{i\in I} u_i) \wedge \left(\wedge_{j\in J} u'_j\right) \longmapsto (\wedge_{i\in I} v_i) \otimes (\wedge_{j\in J} w_j).$$

$\square$

Remark 2.1.3. The determinant functor is defined by Knudsen and Mumford in [48] to take values in the category of graded invertible modules as a resolution of a certain 'sign ambiguity' caused by isomorphisms of the form (2.1). Such a 'sign ambiguity' must be resolved in order to formulate precise leading term conjectures in the equivariant setting (see [13, Rem. 9]). However, for simplicity, we may sometimes suppress the explicit reference to the rank data from the notations in this thesis but readers are reminded that the 'gradings' will always be taken in account.

One important property of the determinant functor (as established in [48, Th. 1]) is that the module $\mathrm{Det}_R(C^\bullet)$ only depends on the complex $C^\bullet$ up to a quasi-isomorphism. Hence, this allows us to extend the construction of determinant modules to the following category.

Definition 2.1.4. A complex $C^\bullet$ of R-modules is perfect if it is quasi-isomorphic to a bounded complex $P^\bullet$ of finitely generated projective modules. For any such complex $C^\bullet$, define its determinant by setting $\mathrm{Det}_R(C^\bullet) := \mathrm{Det}_R(P^\bullet)$.

In particular, the above definition does not depend on the choice of $P^\bullet$. If we write $D^{\mathrm{p}}(R)$ for the full triangulated subcategory of $D(R)$ comprising complexes that are perfect, then the determinant functor factors through $D^{\mathrm{p}}(R)$. For any complex in this category, one can associate a natural notion of an 'Euler characteristic' to it.

Definition 2.1.5. Let $C^\bullet$ be a perfect complex of R-modules. If $P^\bullet = (P^n)_{n\in\mathbb{Z}}$ is a bounded complex of finitely generated projective modules that represents $C^\bullet$ in $D^{\mathrm{p}}(R)$, then the Euler characteristic of $C^\bullet$ is defined to be the element $\sum_{n\in\mathbb{Z}}(-1)^n \mathrm{Det}_R(P^n)$ in $K_0(R)$.

We also note that the isomorphism in Lemma 2.1.2(iii) can be naturally extended as follows. For any exact triangle in $D^{\mathrm{p}}(R)$ of the form $C_1^\bullet \to C_2^\bullet \to C_3^\bullet \to$, there is a canonical isomorphism of graded invertible modules

$$\mathrm{Det}_R(C_2^\bullet) \cong \mathrm{Det}_R(C_1^\bullet) \otimes \mathrm{Det}_R(C_3^\bullet).$$

Finally we record a convenient result that allows one to pass from the determinant of a perfect complex to that of its cohomology modules.

Lemma 2.1.6. Let $C^\bullet$ be a perfect complex of R-modules. If for each $i \in \mathbb{Z}$ the module $H^i(C^\bullet)$ is finitely generated and projective over R, then there is a canonical isomorphism

$$\mathrm{Det}_R(C^\bullet) \xrightarrow{\sim} \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_R^{(-1)^i}(H^i(C^\bullet)).$$

Proof. This is proven by using Lemma 2.1.2 (iii) repeatedly. See [48, Th. 2, Rem. b] for details. □

2.2 Higher Fitting ideals

In this section, we recall the notion of (higher) Fitting ideals. A standard reference for material presented here is [57]. At the outset, we let R be a commutative Noetherian unital ring and M be a finitely generated R-module. In this setting, there is an exact sequence of R-modules of the form

$$R^m \xrightarrow{\phi} R^n \to M \to 0. \tag{2.2}$$

An exact sequence of the form (2.2) is called a presentation of M. In the case when $m = n$, we say that M has a quadratic presentation. Let A be a $n \times m$ matrix representing the R-linear morphism ϕ in the sequence (2.2).

Definition 2.2.1. For $0 \leq i < n$, the i-th Fitting ideal of M, denoted by $\mathrm{Fit}^i_R(M)$, is the ideal of R generated by all the $(n-i) \times (n-i)$ minors of the matrix A. For $i \geq n$, we simply set $\mathrm{Fit}^i_R(M) = R$.

Remark 2.2.2.

(i) Note that the above definition is independent of the choice of presentation of M and hence of the matrix A (see [57, Ch. 3, Thm. 1]).

(ii) Let $R \to R'$ be a ring homomorphism. Because the tensor product functor $- \otimes_R R'$ is right-exact, by applying this functor to the sequence (2.2) we immediately obtain the equality $\mathrm{Fit}^i_{R'}(M \otimes_R R') = \mathrm{Fit}^i_R(M) \otimes_R R'$.

Here we record some useful properties of Fitting ideals. By Laplace's co-factor expansion of determinants, one can immediately deduce that there is an ascending sequence of ideals

$$\mathrm{Fit}^0_R(M) \subseteq \mathrm{Fit}^1_R(M) \subseteq \mathrm{Fit}^2_R(M) \subseteq ... \subseteq \mathrm{Fit}^n_R(M) = R.$$

Here we collect some useful results on Fitting ideals.

Lemma 2.2.3. Let M and N be finitely generated R-modules.

(i) $\mathrm{Fit}^0_R(M) \subseteq \mathrm{Ann}_R(M)$. In particular, if $M = R/I$ is a cyclic R-module, then one has $\mathrm{Fit}^0_R(M) = \mathrm{Ann}_R(M) = I$.

(ii) If there exists a short exact sequence of R-modules of the form $0 \to N \to M \to R^r \to 0$, then for any $a \geq r$ one has $\mathrm{Fit}^a_R(M) = \mathrm{Fit}^{a-r}_R(N)$.

(iii) Assume $N \subseteq M$ and that M/N has a quadratic presentation. Then one has

$$\mathrm{Fit}^0_R(M) = \mathrm{Fit}^0_R(N)\,\mathrm{Fit}^0_R(M/N).$$

Proof. (i) is well-known. (ii) and (iii) are proven in [16, Lem. 7.2(iii), (iv)]. □

2.3 Admissible complexes

In this section, we will define the category of 'admissible complexes' which will be studied in detail in this thesis. We will give examples of objects in this category that arise naturally from the arithmetic context in the next section. At the outset, we fix a Dedekind domain R of characteristic 0 with field of fractions F and a commutative R-order $\mathfrak{A}$ that spans a separable F-algebra $A := F \otimes_R \mathfrak{A}$.

2.3.1 The key definitions

Definition 2.3.1. The category $D^{\mathrm{a}}(\mathfrak{A})$ of 'admissible perfect complexes of $\mathfrak{A}$-modules' is defined to be the full subcategory of $D(\mathfrak{A})$ comprising complexes $C = (C^i)_{i\in\mathbb{Z}}$ that satisfy the following four conditions:

(ad$_1$) C is an object of $D^{\mathrm{p}}(\mathfrak{A})$;

(ad$_2$) the Euler characteristic of $A \otimes_{\mathfrak{A}} C$ in the Grothendieck group $K_0(A)$ vanishes;

(ad$_3$) C is acyclic outside degrees one, two and three;

(ad$_4$) $H^1(C)$ is R-torsion-free.

An object of $D^{\mathrm{a}}(\mathfrak{A})$ that is acyclic outside degrees one and two is said to be a 'strictly admissible perfect complex of $\mathfrak{A}$-modules' and write $D^{\mathrm{s}}(\mathfrak{A})$ to denote the full subcategory of $D^{\mathrm{a}}(\mathfrak{A})$ comprising such complexes.

Remark 2.3.2. Since the F-algebra A is a finite product of fields, the assumptions (ad$_2$) and (ad$_3$) combine to imply there is an isomorphism of A-modules

$$A \otimes_{\mathfrak{A}} H^2(C) \cong H^2(A \otimes_{\mathfrak{A}} C) \cong H^1(A \otimes_{\mathfrak{A}} C) \oplus H^3(A \otimes_{\mathfrak{A}} C) \cong A \otimes_{\mathfrak{A}} (H^1(C) \oplus H^3(C)).$$

Remark 2.3.3. These categories have already been studied in the earlier work of Barrett and Burns in [2]. The work of Burns and Macias Castillo in [21] generalised the setup to the non-commutative case. In this thesis, we further study objects of these categories to derive stronger annihilation and structural results on their cohomology groups in the case when $\mathfrak{A}$ is commutative.

Remark 2.3.4. Let $\mathfrak{p}$ be a prime ideal of R and $A'_{\mathfrak{p}}$ a local component of the semi-local algebra $A_{\mathfrak{p}}$. Then, for each complex C in $D^{\mathrm{a}}(\mathfrak{A})$, respectively in $D^{\mathrm{s}}(\mathfrak{A})$, the complex $C'_{\mathfrak{p}} := \mathfrak{A}'_{\mathfrak{p}} \otimes_{\mathfrak{A}} C$

belongs to $D^{\mathrm{a}}(\mathfrak{A}'_{\mathfrak{p}})$, respectively $D^{\mathrm{s}}(\mathfrak{A}'_{\mathfrak{p}})$. In particular, since each finitely generated torsion-free $A'_{\mathfrak{p}}$-module has finite projective dimension if and only if it is free, a standard argument of homological algebra combines with the assumptions (ad$_1$), (ad$_3$) and (ad$_4$) to imply that for each C in $D^{\mathrm{a}}(\mathfrak{A})$ the complex $C'_{\mathfrak{p}}$ is isomorphic in $D(\mathfrak{A}'_{\mathfrak{p}})$ to a complex of $\mathfrak{A}'_{\mathfrak{p}}$-modules

$$P^1 \xrightarrow{d^1} P^2 \xrightarrow{d^2} P^3 \tag{2.3}$$

in which each module is both finitely generated and free and the first term is placed in degree one. If C belongs to $D^{\mathrm{s}}(\mathfrak{A})$, then one can in addition take the module P^3 to be zero.

2.3.2 Useful constructions

We now record two important and new ways in which admissible complexes give rise to new families of admissible complexes. (see §1.9 for the direction of shifting of complexes)

Lemma 2.3.5. Assume we are given the following data:

(a) an object C of $D^{\mathrm{a}}(\mathfrak{A})$, and

(b) a finitely generated projective $\mathfrak{A}$-module P and a homomorphism of $\mathfrak{A}$-modules

$$\theta^i : P \to H^i(C)$$

for $i = 1$ and $i = 2$ where θ^1 is both injective and such that $\mathrm{cok}(\theta^1)$ is R-torsion-free.

Then there is a exact triangle in $D(\mathfrak{A})$ of the form

$$P[-1] \oplus P[-2] \xrightarrow{\theta} C \to D \to P[0] \oplus P[-1] \tag{2.4}$$

in which θ is the unique morphism with $H^i(\theta) = \theta^i$ for $i = 1, 2$ and D belongs to $D^{\mathrm{a}}(\mathfrak{A})$.

Proof. Since P is projective, the natural map

$$\mathrm{Hom}_{D^{\mathrm{p}}(\mathfrak{A})}(P[-1]\oplus P[-2],C)\to \mathrm{Hom}_{\mathfrak{A}}(P,H^1(C))\oplus \mathrm{Hom}_{\mathfrak{A}}(P,H^2(C))$$

is bijective and hence there exists a unique morphism θ with the stated properties.

Choose D to be any complex that lies in an exact triangle (2.4). Then it is clear that D belongs to $D^{\mathrm{p}}(\mathfrak{A})$ and that the Euler characteristic of $A\otimes_{\mathfrak{A}} D$ in $K_0(A)$ vanishes (since this is true for both C and $P[-1]\oplus P[-2]$). In addition, the long exact cohomology sequence of (2.4) has the form

$$P \xrightarrow{\theta^1} H^1(C)\to H^1(D)\to P \xrightarrow{\theta^2} H^2(C)\to H^2(D)\to 0\to H^3(C)\to H^3(D)\to 0 \qquad (2.5)$$

and so implies immediately that D is acyclic outside degrees one, two and three and combines with the assumption that θ^1 is injective and that $\mathrm{cok}(\theta^1)$ is R-torsion-free to imply that $H^1(D)$ is also R-torsion-free. ☐

In the next result, assume we are given a homomorphism of R-orders $\mathfrak{B}\to\mathfrak{A}$ and use it to regard each $\mathfrak{A}$-module M as a $\mathfrak{B}$-module. We then regard the linear dual $\mathrm{Hom}_{\mathfrak{B}}(M,\mathfrak{B})$ of any such M as an $\mathfrak{A}$-module by the rule $a(f)(m):=f(a(m))$ for each a in $\mathfrak{A}$, f in $\mathrm{Hom}_{\mathfrak{B}}(M,\mathfrak{B})$ and m in M.

Recall that a R-order $\mathfrak{B}$ is Gorenstein if $\mathrm{Hom}_R(\mathfrak{B},R)$ is projective as a $\mathfrak{B}$-module (see [28, §37] for other equivalent definitions). We note that if $\mathfrak{B}$ is Gorenstein and X is a finitely generated $\mathfrak{B}$-module which is R-torsion-free, then one has $\mathrm{Ext}^1_{\mathfrak{B}}(X,\mathfrak{B})=0$ (see [23, (56)]).

We say that $\mathfrak{A}$ is 'everywhere locally Gorenstein relative to $\mathfrak{B}$' if for each prime ideal $\mathfrak{p}$ of R the $R_{\mathfrak{p}}\otimes_R\mathfrak{A}$-module $R_{\mathfrak{p}}\otimes_R \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{A},\mathfrak{B})$ is free of rank one.

Lemma 2.3.6. Let $\mathfrak{B}\to\mathfrak{A}$ be a homomorphism of R-orders that satisfies all of the following three conditions.

(a) $\mathfrak{A}$ is a projective $\mathfrak{B}$-module,

(b) $\mathfrak{A}$ is everywhere locally Gorenstein relative to $\mathfrak{B}$,

(c) $\mathfrak{B}$ is Gorenstein.

Then the functors $C \mapsto \mathrm{R}\,\mathrm{Hom}_{\mathfrak{B}}(C, \mathfrak{B}[-4])$ (resp. $C \mapsto \mathrm{R}\,\mathrm{Hom}_{\mathfrak{B}}(C, \mathfrak{B}[-3])$) send any object of the category $D^{\mathrm{a}}(\mathfrak{A})$ (resp. $D^{\mathrm{s}}(\mathfrak{A})$) to itself.

Proof. For any $\mathfrak{A}$-module Q we set $Q^* := \mathrm{Hom}_{\mathfrak{B}}(Q, \mathfrak{B})$.

Then the assumption (b) implies that for any finitely generated projective $\mathfrak{A}$-module Q the $\mathfrak{A}$-module Q^* is finitely generated and projective. This implies that the category $D^{\mathrm{p}}(\mathfrak{A})$ is preserved by the functor $C \mapsto C^* := \mathrm{R}\,\mathrm{Hom}_{\mathfrak{B}}(C, \mathfrak{B})$.

To compute the cohomology groups of C^* we claim first that for any finitely generated $\mathfrak{A}$-module N and for all integers $j > 1$ the group $\mathrm{Ext}^j_{\mathfrak{B}}(N, \mathfrak{B})$ vanishes. To prove this we note that any exact sequence of $\mathfrak{A}$-modules of the form

$$0 \to N' \to \mathfrak{A}^d \to N \to 0,$$

induces, as a consequence of assumption (a), an isomorphism of the form $\mathrm{Ext}^j_{\mathfrak{B}}(N, \mathfrak{B}) \cong \mathrm{Ext}^{j-1}_{\mathfrak{B}}(N', \mathfrak{B})$. Thus, since each such module N' is torsion-free over R, the vanishing of $\mathrm{Ext}^j_{\mathfrak{B}}(N, \mathfrak{B})$ can be reduced, by a downward induction on j, to the vanishing of $\mathrm{Ext}^1_{\mathfrak{B}}(N, \mathfrak{B})$ for every finitely generated $\mathfrak{B}$-module N that is torsion-free over R. It is then enough to note that the latter condition is equivalent to condition (c) (cf. [28, §37] and [30, Prop. 6.1]).

Then, since the groups $\mathrm{Ext}^j_{\mathfrak{B}}(H^i(C), \mathfrak{B})$ vanish for all integers $j > 1$ and all integers i, the universal coefficient theorem (see [79, Th. 3.6.5]) implies that in each degree i there is a canonical short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathfrak{B}}(H^{1-i}(C)_{\mathrm{tor}}, \mathfrak{B}) \to H^i(C^*) \to H^{-i}(C)^* \to 0. \tag{2.6}$$

Here we have also used the fact that the tautological exact sequence

$$0 \to H^{1-i}(C)_{\mathrm{tor}} \to H^{1-i}(C) \to H^{1-i}(C)_{\mathrm{tf}} \to 0$$

combines with assumption (c) to induce an isomorphism

$$\mathrm{Ext}^1_{\mathfrak{B}}(H^{1-i}(C), \mathfrak{B}) \cong \mathrm{Ext}^1_{\mathfrak{B}}(H^{1-i}(C)_{\mathrm{tor}}, \mathfrak{B}).$$

By using the sequence (2.6) to compute the cohomology groups of $C^*[-3]$ and $C^*[-4]$ one finds that the functors $C \mapsto C^*[-4]$ and $C \mapsto C^*[-3]$ respectively preserve the categories $D^{\mathrm{a}}(\mathfrak{A})$ and $D^{\mathrm{s}}(\mathfrak{A})$, as claimed. $\square$

Example 2.3.7. Here we give examples in which the hypotheses of Lemma 2.3.6 are satisfied.
(i) Let $\mathfrak{B} \to \mathfrak{A}$ be the tautological homomorphism $R \to \mathfrak{A}$. Then assumptions (a) and (c) are automatically satisfied and assumption (b) is satisfied if, for example, $\mathfrak{A}$ is a direct factor of the group ring $R[\Gamma]$ for a finite abelian group Γ.
(ii) If $\mathfrak{B} \to \mathfrak{A}$ is the identity map $\mathfrak{A} \to \mathfrak{A}$, then the conditions (a), (b) and (c) are all satisfied if and only if $\mathfrak{A}$ is Gorenstein.

2.3.3 Controlling the cohomology of highest degree

In this section we describe a new and natural construction that every admissible complex gives rise to natural families of strictly admissible complexes.

To do this, assume we are given an object C of $D^{\mathrm{a}}(\mathfrak{A})$ and we write $e_0 = e_{C,0}$ for the sum of all primitive idempotents e of A which are such that the module $e(F \cdot H^3(C))$ vanishes. We also write $\mathfrak{A}_0 = \mathfrak{A}_{C,0}$ for the order $\mathfrak{A}e_0$ and C_0 for the object $\mathfrak{A}_0 \otimes^{\mathbb{L}}_{\mathfrak{A}} C$ of $D(\mathfrak{A}_0)$.

We can now state the main result to be proved in this section.

Proposition 2.3.8. Let C be an object of $D^{\mathrm{a}}(\mathfrak{A})$.

Then, for each prime ideal $\mathfrak{p}$ of R, there exists a set of generators $\mathcal{F}_{\mathfrak{p}}$ of the $\mathfrak{A}_{\mathfrak{p}}$-module $\mathrm{Fit}^0_{\mathfrak{A}}(H^3(C)) \cdot \mathfrak{A}_{0,\mathfrak{p}}$ that are each invertible in $A_{0,\mathfrak{p}}$ and are such that for each x in $\mathcal{F}_{\mathfrak{p}}$ there is a complex C_x in $D^{\mathrm{s}}(\mathfrak{A}_{0,\mathfrak{p}})$ with all of the following properties:

(i) $H^1(C_x) = H^1(C_{0,\mathfrak{p}})$.

(ii) $H^2(C_x)$ contains $H^2(C_{0,\mathfrak{p}})$ as a submodule of finite index and the quotient module $H^2(C_x)/H^2(C_{0,\mathfrak{p}})$ is annihilated by x.

(iii) $e_0 \cdot \mathrm{Det}_{\mathfrak{A}}(C)_{\mathfrak{p}} = x \cdot \mathrm{Det}_{\mathfrak{A}_{0,\mathfrak{p}}}(C_x)$.

Proof. If necessary, we can replace R by $R_{\mathfrak{p}}$, $\mathfrak{A}$ by a local component $A'_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ and C by $\mathfrak{A}'_{\mathfrak{p}} \otimes_{\mathfrak{A}} C$ to assume in the sequel that $\mathfrak{A}$ is local.

Then, as C belongs to $D^{\mathrm{a}}(\mathfrak{A})$, the observation in Remark 2.3.4 implies that C is represented by a complex of finitely generated free $\mathfrak{A}$-modules of the form (2.3).

Having chosen such a representative, for each $i \in \{1, 2, 3\}$ we write s_i for the rank of the $\mathfrak{A}$-module P^i and fix a basis $\{x^i_j\}_{1\le j\le s_i}$ of this module. We assume, as we may, that $s_2 \ge s_3$ and we choose an $s_3 \times s_3$ minor $M := (m_{ij})_{1\le i,j\le s_3}$ of the matrix of d^2 with respect to these bases.

Then for each integer j with $1 \le j \le s_3$ the element $\sum_{i=1}^{i=s_3} m_{ji} \cdot x^3_i$ belongs to $\mathrm{Im}(d^2)$ and we fix a pre-image b_j in P^2 of it under d^2. We also fix a multiple n of $|H^3(C)|$ and an element c_j of P^2 with $d^2(c_j) = n \cdot x^3_j$.

We write $\phi = \phi_{\{b_\bullet\},\{c_\bullet\}}$ for the homomorphism $P^3 \to P^2$ of $\mathfrak{A}$-modules that sends each element x_j to $b_j + c_j$ and consider the following commutative diagram

$$\begin{array}{ccccc} P^1 & \xrightarrow{d^1} & P^2 & \xrightarrow{d^2} & P^3 \\ {\scriptstyle (\mathrm{id},0)}\big\downarrow & & \big\| & & \\ P^1 \oplus P^3 & \xrightarrow{(d^1,\phi)} & P^2. & & \end{array} \tag{2.7}$$

We write C_ϕ for the complex given by the lower row of the diagram, with the first term placed in degree one. Then the diagram constitutes a morphism $\theta : C \to C_\phi$ of complexes of $\mathfrak{A}$-modules, the mapping cone of which is the upper complex in the following morphism of complexes

$$\begin{tikzcd} P^1 \arrow[r, "(\mathrm{id},0,d^1)"] & (P^1 \oplus P^3) \oplus P^2 \arrow[r, "(((d^1,\phi),0)-(0,\mathrm{id}),-d^2)"] \arrow[d, "(0,\mathrm{id},0)"'] & P^2 \oplus P^3 \arrow[d, "(d^2,-\mathrm{id})"] \\ & P^3 \arrow[r, "d^2\circ\phi"] & P^3. \end{tikzcd} \tag{2.8}$$

Here the first term in the upper complex is placed in degree zero and an explicit check shows that the two vertical arrows constitute a quasi-isomorphism of complexes.

Writing $\mathrm{cone}(\theta)'$ for the lower complex in (2.8), one therefore obtains an exact triangle

$$C \to C_\phi \to \mathrm{cone}(\theta)' \to C[1]$$

in $D(\mathfrak{A})$ and hence an associated long exact sequence of cohomology

$$0 \to H^1(C) \to H^1(C_\phi) \to \ker(d^2 \circ \phi) \to H^2(C) \to H^2(C_\phi) \to \mathrm{cok}(d^2 \circ \phi) \to H^3(C) \to 0.$$

Now, with respect to the given basis $\{x_j^3\}_{1\le j\le s_3}$ of P^3, the matrix of $d^2 \circ \phi$ is $M + n \cdot I$, with I the $s_3 \times s_3$ identity matrix. In particular, by choosing n large enough we can ensure both that $d^2 \circ \phi$ is injective, and hence that $\mathrm{cok}(d^2 \circ \phi)$ is finite and $\mathrm{cone}(\theta)'$ is isomorphic in $D(\mathfrak{A})$ to $\mathrm{cok}(d^2 \circ \phi)[-2]$, and that

$$\det(M + n \cdot I) \equiv \det(M) \;\text{ modulo }\; \mathfrak{p} \cdot |H^3(C)| \cdot \mathrm{Fit}^0_{\mathfrak{A}}(H^3(C)).$$

This last observation shows that as we vary the choice of minor M, the elements $\det(d^2 \circ \phi) = \det(M + n \cdot I)$ are invertible in A and constitute a set $\mathcal{F}$ of generators of $\mathrm{Fit}^0_{\mathfrak{A}}(H^3(C))$.

For each minor M we then set $x := \det(d^2 \circ \phi)$ and $C_x := C_\phi$ and $Q_x := \mathrm{cok}(d^2 \circ \phi)$.

With these choices, claim (i) and (ii) are clear. To prove claim (iii) we note that the above construction gives an exact triangle

$$C \to C_x \to Q_x[-2] \to C[1]$$

in $D^{\mathrm{p}}(\mathfrak{A})$ and hence gives rise to an equality of invertible $\mathfrak{A}$-modules

$$\mathrm{Det}_{\mathfrak{A}}(C_x) = \mathrm{Det}_{\mathfrak{A}}(C) \cdot \mathrm{Det}_{\mathfrak{A}}(Q_x[-2]).$$

It is thus enough to note that the exact sequence of $\mathfrak{A}$-modules

$$0 \to P^3 \xrightarrow{d^2 \circ \phi} P^3 \to Q_x \to 0 \tag{2.9}$$

implies that (the ungraded part of) $\mathrm{Det}_{\mathfrak{A}}(Q_x[-2])$ is generated over $\mathfrak{A}$ by the inverse of the invertible element $\det(d^2 \circ \phi) =: x$. $\square$

2.4 Arithmetic examples

In this section, we give arithmetic examples of (strictly) admissible complexes that will be discussed in further detail in this thesis.

At the outset we fix a finite abelian extension F/k of number fields and set $G := \mathrm{Gal}(F/k)$. We also fix a finite set of places S of k that contains all places that are either archimedean or ramify in F/k and an auxiliary finite set of places T in k that is disjoint from S. We write $\mathcal{O}_{F,S}$ for the ring of S_F-integers in F.

2.4.1 Weil-étale cohomology for $\mathbb{G}_m$

In [16], Burns, Kurihara and Sano describe a canonical 'Weil-étale cohomology' complex $R\Gamma_T((\mathcal{O}_{F,S})_{\mathcal{W}}, \mathbb{G}_m)$ for the multiplicative group $\mathbb{G}_m$ over F. Instead of recalling the detailed construction of this complex, we will record its relevant properties here.

To do this we write $\mathcal{O}_{F,S,T}^\times$ for the (finite index) subgroup of $\mathcal{O}_{F,S}^\times$ defined by

$$\mathcal{O}_{F,S,T}^\times := \{a \in \mathcal{O}_{F,S}^\times : a \equiv 1 \pmod{w} \text{ for all } w \in T_F\}$$

and $\mathrm{Sel}_S^T(F)^{\mathrm{tr}}$ for the transpose integral Selmer module defined in [16, Def. 2.6] (where it was denoted $\mathcal{S}_{S,T}^{\mathrm{tr}}(\mathbb{G}_{m/F})$). We recall that it is shown in [16, Rem. 2.7] that the module $\mathrm{Sel}_S^T(F)^{\mathrm{tr}}$ lies in a canonical short exact sequence

$$0 \to \mathrm{Cl}_S^T(F) \to \mathrm{Sel}_S^T(F)^{\mathrm{tr}} \to X_{F,S} \to 0 \tag{2.10}$$

where $\mathrm{Cl}_S^T(F)$ is the ray class group of $\mathcal{O}_{F,S}$ modulo the product of all places in T_F (to be more specific, it is the quotient of the group of fractional ideals of F whose supports are coprime to all places in $S_F \cup T_F$ by the subgroup of principal ideals with a generator congruent to 1 modulo all places in T_F) and $X_{F,S}$ is the kernel of the natural morphism $\bigoplus_{w \in S_F} \mathbb{Z} \to \mathbb{Z}$. Note that in the case when T is empty, we will suppress it from any notation.

Remark 2.4.1. Recall from [16] that the 'S-relative T-trivialized Selmer group' $\mathrm{Sel}_S^T(F)$ for $\mathbb{G}_m$ over F is the cokernel of the canonical homomorphism $\prod_w \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F_T^\times, \mathbb{Z})$. Here w runs over all places of F that are not in $S_F \cup T_F$, $F_T^\times$ is the subgroup of $F^\times$ comprising elements a with $\mathrm{ord}_w(a-1) > 0$ for all places w of T_F and the homomorphism sends each element $(x_w)_w$ to the map $a \mapsto \sum_w \mathrm{ord}_w(a) x_w$. This group is a natural analogue for $\mathbb{G}_m$ of the 'integral Selmer groups' that are defined for abelian varieties by Mazur and Tate in [52].

In particular, it lies in an exact sequence [16, Prop. 2.2]

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}_S^T(F), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Sel}_S^T(F) \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{F,S,T}^{\times}, \mathbb{Z}) \to 0. \tag{2.11}$$

We also note that the module $\mathrm{Sel}_S^T(F)^{\mathrm{tr}}$ can be regarded as the canonical transpose of $\mathrm{Sel}_S^T(F)$ in the sense of Jannsen's homotopy theory of modules [39].

Proposition 2.4.2. Let $C_{F,S,T} := R\Gamma_T((\mathcal{O}_{F,S})_{\mathcal{W}}, \mathbb{G}_m)[-1]$. If the group $\mathcal{O}_{F,S,T}^{\times}$ is $\mathbb{Z}$-torsion-free, then $C_{F,S,T}$ is an object in $D^{\mathrm{s}}(\mathbb{Z}[G])$ and there are identifications $H^1(C_{F,S,T}) = \mathcal{O}_{F,S,T}^{\times}$ and $H^2(C_{F,S,T}) = \mathrm{Sel}_S^T(F)^{\mathrm{tr}}$.

Proof. The results of [16, §2.2] imply that the complex $R\Gamma_T((\mathcal{O}_{F,S})_{\mathcal{W}}, \mathbb{G}_m)[-1]$ has all of the required properties. $\square$

Remark 2.4.3. Suppose S is large enough such that $\mathrm{Cl}_S(F)$ vanishes. In this case, it was remarked in [16, Prop. 2.9] that the complex $R\Gamma((\mathcal{O}_{F,S})_{\mathcal{W}}, \mathbb{G}_m)$ defines an element in $\mathrm{Ext}^2_{\mathbb{Z}[G]}(X_{F,S}, \mathcal{O}_{F,S}^{\times})$ and this element coincides with the 'Tate's fundamental class' constructed in [75] by class field theory. In the literature, the Tate's fundamental class and its associated Yoneda 2-extension play a key role in the studies of Equivariant Tamagawa Number Conjecture for Tate motives and its relation with the (refined) abelian Stark conjectures. See [8] and [77] for details.

2.4.2 Compactly supported étale cohomology

Another rich source of admissible complexes is the compactly supported cohomology complexes which have been used in Kato's reformulation of his celebrated 'Tamagawa Number Conjecture' with Bloch (see [45, §2.1]). In this section, we recall their construction and some important properties. We will adopt the notation $F/k, S, G$ from the beginning of §2.4. We futher fix a rational prime p and assume S also contains the p-adic places of k. We write $G_k := \mathrm{Gal}(k^c/k)$.

If R denotes either $\mathcal{O}_{k,S}, k$ or the completion k_v of k at a place v and $\mathcal{F}$ is an étale (pro-) sheaf on $\mathrm{Spec}(R)$, then we abbreviate the complex $R\Gamma_{\text{ét}}(\mathrm{Spec}(R), \mathcal{F})$ and in each degree a the group $H^a_{\text{ét}}(\mathrm{Spec}(R), \mathcal{F})$ to $R\Gamma(R, \mathcal{F})$ and $H^a(R, \mathcal{F})$ respectively.

Definition 2.4.4. For any étale (pro-)sheaf $\mathcal{F}$ on $\mathrm{Spec}(\mathcal{O}_{k,S})$, define the compact support cohomology complex $R\Gamma_c(\mathcal{O}_{k,S}, \mathcal{F})$ by means of the exact triangle

$$R\Gamma_c(\mathcal{O}_{k,S}, \mathcal{F}) \to R\Gamma(\mathcal{O}_{k,S}, \mathcal{F}) \to \bigoplus_{v \in S} R\Gamma(k_v, \mathcal{F}) \to \tag{2.12}$$

where the second arrow is the direct sum of the natural localisation morphisms. For each integer a we set $H^a_c(\mathcal{O}_{k,S}, \mathcal{F}) := H^a(R\Gamma_c(\mathcal{O}_{k,S}, \mathcal{F}))$.

Now assume we are given a pair $(\mathfrak{A}, T)$ comprising a $\mathbb{Z}_p$-order $\mathfrak{A}$ that spans a separable $\mathbb{Q}_p$-algebra A and a continuous $\mathbb{Z}_p[G_k]$-module T that is unramified outside S and is endowed with a commuting action of $\mathfrak{A}$ with respect to which it is a projective module. We also set $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ and $W := V/T$.

Example 2.4.5. The pair described above occurs naturally in several occasions in arithmetic. For example, if we let T be a finitely generated free $\mathbb{Z}_p$-module that is endowed with a continuous action of G_k unramified outside S (usually referred as a 'p-adic representation' in the literature) and set $\mathfrak{A} := \mathbb{Z}_p[G]$, then the tensor product $T_F := \mathfrak{A} \otimes_{\mathbb{Z}_p} T$ becomes a module over $\mathfrak{A} \times \mathbb{Z}_p[G_k]$ in the following way: the action of $\mathfrak{A}$ is induced by letting G act via multiplication on the left and each $u \in G_k$ acts as $x \otimes_{\mathbb{Z}_p} t \mapsto x\bar{u}^{-1} \otimes_{\mathbb{Z}_p} u(t)$ for $x \in \mathfrak{A}$ and $t \in T$ where $\bar{u}$ denotes the image of u in $G_{F/k}$. Then, with respect to this action T_F is both unramified outside S and a free $\mathfrak{A}$-module.

In the rest of this section, we abbreviate $C(T) := R\Gamma_c(\mathcal{O}_{k,S}, T)$.

Proposition 2.4.6. For any data $(\mathfrak{A}, T)$ as above, the complex $C(T)$ belongs to $D^{\mathrm{a}}(\mathfrak{A})$.

Proof. This is well-known. See [2, Lem. 3.1] for example. □

Examples 2.4.7.

(i) Set $T = \mathbb{Z}_p$. Then it is shown in [12, Prop. 3.3] that there is an isomorphism in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ between

$$C_{F,S} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong R\,\mathrm{Hom}_{\mathbb{Z}_p}(C(T), \mathbb{Z}_p)[-2]$$

where $C_{F,S} := R\Gamma((\mathcal{O}_{F,S})_{\mathcal{W}}, \mathbb{G}_m)$ is the Weil-étale cohomology complex from §2.4.1.

(ii) Set $T = \mathbb{Z}_p(1)$. In this case, the second and third cohomology groups of $C(T)$ were explicitly computed in [5, (10)] as follows. If we write $M_{F,S}$ for the maximal abelian pro-p extension of F in F^c that is unramified outside S and set $G_{F,S} := \mathrm{Gal}(M_{F,S}/F)$, then one has $H^2(C(T)) = G_{F,S}$ and $H^3(C(T)) = \mathbb{Z}_p$.

(iii) Set $T = \mathbb{Z}_p(r)$ for any integer r. Then the Artin-Verdier duality (of the form [14, (6)]) implies that there is an exact triangle in $D(\mathbb{Z}_p[G])$ of the form

$$R\Gamma_c(\mathcal{O}_{F,S}, T) \to R\,\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_{F,S}, T^*(1)), \mathbb{Z}_p)[-3] \to \bigoplus_{w \in S_\infty(F)} H^0(F_w, T)[0] \to .$$

By taking the dual, we obtain the exact triangle

$$R\Gamma(\mathcal{O}_{F,S}, T^*(1))[1] \to R\,\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{F,S}, T), \mathbb{Z}_p)[-2] \to (\bigoplus_{w \in S_\infty(F)} H^0(F_w, T))^*[-1] \to .$$

If we then set $C^\bullet := R\Gamma(\mathcal{O}_{F,S}, T^*(1))[1] \oplus (\bigoplus_{w\in S_\infty(F)} H^0(F_w, T))^*[-1]$, the above triangle implies that there is an isomorphism $C^\bullet \cong R\,\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{F,S}, T), \mathbb{Z}_p)[-2]$ in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$. In [18], the complex $C^\bullet$ is shown to have a role in the connections between the 'generalised Stark elements' with relevant cases of the Equivariant Tamagawa Number Conjecture. This will be discussed in further detail in §4.2.

2.4.3 Selmer and Tate-Shafarevich groups à la Bloch-Kato

In this subsection, we recall the definitions of finite support cohomology, Selmer and Tate-Shafarevich groups of Bloch and Kato given in [3]. We adopt the notation from §2.4.2. In addition, for each place v, we write I_v for the inertia subgroup of G_k.

Definition 2.4.8.

(i) The local compact support cohomology group is defined by

$$H^1_f(k_v, V) := \begin{cases} 0 & \text{if } v \in S_\infty, \\ \ker(H^1(k_v, V) \to H^1(k_v, V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})) & \text{if } v \in S_p, \\ \ker(H^1(k_v, V) \to H^1(I_v, V)) & \text{otherwise.} \end{cases} \tag{2.13}$$

Furthermore, define $H^1_f(k_v, T)$ (resp. $H^1_f(k_v, W)$) to be the pre-image (resp. image) of $H^1_f(k_v, V)$, under the natural map $H^1(k_v, T) \to H^1(k_v, V)$ (resp. $H^1(k_v, V) \to H^1(k_v, W)$).

(ii) If $\mathcal{F}$ denotes either of T, V or W, then the global finite support cohomology group $H^1_f(k, \mathcal{F})$ of Bloch and Kato is defined by the natural exact sequence

$$0 \to H^1_f(k, \mathcal{F}) \xrightarrow{\subseteq} H^1(\mathcal{O}_{k,\Sigma}, \mathcal{F}) \to \bigoplus_{v \in S} \frac{H^1(k_v, \mathcal{F})}{H^1_f(k_v, \mathcal{F})}, \tag{2.14}$$

and hence there is an induced localisation map $\lambda_{\mathcal{F}} : H^1_f(k, \mathcal{F}) \to \bigoplus_{v \in S} H^1_f(k_v, \mathcal{F})$.

(iii) The Bloch-Kato Selmer and Tate-Shafarevic groups $\mathrm{Sel}(T)$ and $\text{Ш}(T)$ of T are defined to be equal to $H^1_f(k, W)$ and the cokernel of the natural homomorphism $H^1_f(k, V) \to H^1_f(k, W) = \mathrm{Sel}(T)$ respectively.

Remark 2.4.9. In this remark, we collect some facts about the Bloch-Kato Selmer and Tate-Shafarevic groups from the literature which will be useful in this thesis.

(i) Immediately from the definition, there is a canonical short exact sequence

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} H^1_f(k, T^*(1)) \to \mathrm{Sel}(T^*(1)) \to Ш(T^*(1)) \to 0. \tag{2.15}$$

It was shown in [35, Chap. II, 5.3.5] that $Ш(T^*(1))$ is finite. Hence the above sequence induces an identification of $Ш(T^*(1))$ with $\mathrm{Sel}(T^*(1))_{\mathrm{cotor}}$.

(ii) The main result of Flach [33, Thm. 1] shows that there exists a canonical isomorphism between $Ш(T)$ and $Ш(T^*(1))^\vee$. Furthermore, Flach has also shown in [33, Ex. p.122-123] that there is a canonical identification $Ш(\mathbb{Z}_p(1)_F)$ with $\mathrm{Cl}(F) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Now we will state an important theorem that would give rise to an admissible complex for which the Tate-Shafarevic group will appear as a subquotient of its cohomology.

Proposition 2.4.10. Assume that the diagonal localisation homomorphism

$$H^1(\mathcal{O}_{k,S}, T)_{\mathrm{tor}} \to \bigoplus_{v \in S} H^1(k_v, T)_{\mathrm{tor}}$$

is injective. Then for any homomorphism

$$\phi : \bigoplus_{v \in S_\infty} H^0(G_{\mathbb{C}/\mathbb{R}}, T) \to \bigoplus_{v \in S_p} H^1_f(k_v, T)$$

of $\mathfrak{A}$-modules, the data $(C(T), \phi)$ gives rise via the construction in Lemma 2.3.5 to a canonical object $C_\phi(T)$ of $D^{\mathrm{a}}(\mathfrak{A})$ with the property that $\mathrm{Sel}(T^*(1))^\vee$, and hence also $Ш(T)$ is isomorphic to a subquotient of $H^2(C_\phi(T))$.

Proof. First we note that a comparison between (2.14) and the long exact cohomology sequence of (2.12) with $\mathcal{F} = T$ shows that the localisation homomorphism λ_T fits into an exact sequence

$$H^1_f(k,T) \xrightarrow{\lambda_T} \bigoplus_{v\in S} H^1_f(k_v,T) \xrightarrow{\tilde{\lambda}_T} H^2(C(T)) \to \mathrm{cok}(\tilde{\lambda}_T) \to 0, \tag{2.16}$$

where $\tilde{\lambda}_T$ is induced from the connecting morphism of (2.12). Now for each place v in S, the Pontryagin dual of the tautological exact sequence

$$0 \to H^1_f(k_v,T) \xrightarrow{\subseteq} H^1(k_v,T) \to \frac{H^1(k_v,V)}{H^1_f(k_v,V)}$$

combines with the local duality isomorphism ([64, Thm 1.4.1]) $H^1(k_v,T)^\vee \cong H^1(k_v,W^*(1))$ and the definition of $H^1_f(k_v,W^*(1))$ to imply that $H^1_f(k_v,T)^\vee$ is naturally isomorphic to the quotient $H^1(k_v,W^*(1))/H^1_f(k_v,W^*(1))$. Hence, upon taking the Pontryagin dual of (2.16), and using the Artin-Verdier duality isomorphism (of the form [54, Ch. 3, Cor. 3.4])

$$H^2(C(T))^\vee \cong H^1(\mathcal{O}_{k,S},W^*(1)),$$

one obtains an exact sequence

$$0 \to \mathrm{cok}(\tilde{\lambda}_T)^\vee \to H^1(\mathcal{O}_{k,S},W^*(1)) \xrightarrow{\tilde{\lambda}_T^\vee} \bigoplus_{v\in\Sigma} \frac{H^1(k_v,W^*(1))}{H^1_f(k_v,W^*(1))}$$

in which $\tilde{\lambda}_T^\vee$ identifies with the sum of the natural localisation maps. Since $\mathrm{Sel}(T^*(1))$ is defined to be $\ker(\tilde{\lambda}_T^\vee)$ we thus obtain an isomorphism $\mathrm{cok}(\tilde{\lambda}_T) \cong \mathrm{Sel}(T^*(1))^\vee$ and so (2.16) induces an exact sequence

$$0 \to \mathrm{cok}(\lambda_T) \to H^2(C(T)) \to \mathrm{Sel}(T^*(1))^\vee \to 0. \tag{2.17}$$

Now we set $P := \bigoplus_{v\in S_\infty} H^0(G_{\mathbb{C}/\mathbb{R}}, T)$. To apply Lemma 2.3.5, we must define suitable homomorphisms $\theta^1 : P \to H^1(C(T))$ and $\theta^2 : P \to H^2(C(T))$. To do this, we first note that the exact triangle (2.12) gives rise to a canonical long exact sequence

$$0 \to H^0(\mathcal{O}_{k,S}, T) \xrightarrow{\kappa} \bigoplus_{v\in S} H^0(k_v, T) \xrightarrow{\alpha} H^1(C(T)) \to H^1(\mathcal{O}_{k,S}, T) \xrightarrow{\lambda} \bigoplus_{v\in S} H^1(k_v, T),$$

in which κ is the diagonal inclusion map, λ the diagonal localisation map, α is the connecting homomorphism. This implies that if we define θ^1 to be the restriction of α to $P = \bigoplus_{v\in S_\infty} H^0(G_{k_v}, T)$, then θ^1 is injective and there is an exact sequence

$$0 \to H^0(G_{k,S}, T) \xrightarrow{\kappa'} \bigoplus_{v\in S\setminus S_\infty} H^0(G_{k_v}, T) \to \mathrm{cok}(\theta^1) \to \ker(\lambda) \to 0$$

in which κ' is the diagonal inclusion map. It follows that $\mathrm{cok}(\theta^1)$ is torsion-free if $\mathrm{cok}(\kappa')$ and $\ker(\lambda)$ are both torsion-free. Since T is torsion-free, we have that $\mathrm{cok}(\kappa')$ is torsion-free. Moreover, the (assumed) injectivity of the displayed localisation map in Proposition 2.4.10 implies directly that $\ker(\lambda)$ is torsion-free. Hence, we deduce that $\mathrm{cok}(\theta^1)$ is torsion-free.

In addition, if we define θ^2 to be the composite homomorphism of $\mathfrak{A}$-modules

$$P \to \bigoplus_{v\in S_p} H^1_f(k_v, T) \to \mathrm{cok}(\lambda_T) \to H^2(C(T)),$$

where the first map is the given homomorphism ϕ, the second is the tautological projection and the third is from (2.17), then the exact sequence (2.17) implies that $\mathrm{Sel}(T^*(1))^\vee$, and hence also $\text{Ш}(T) \cong \text{Ш}(T^*(1))^\vee$, is isomorphic to a subquotient of $\mathrm{cok}(\theta^2)$.

Thus, to complete the proof, it suffices to define $C_\phi(T)$ to be the complex D constructed when Lemma 2.3.5 is applied to the data (C, θ^1, θ^2) and to note that the exact sequence (2.5) identifies $H^2(D)$ with $\mathrm{cok}(\theta^2)$. $\square$

2.5 Algebra of dual modules

We now collect all the algebraic results regarding the two kinds of dual that will be considered in this thesis. In the next two sections, we will assume that $\mathfrak{A}$ is Gorenstein.

Definition 2.5.1. For any $\mathfrak{A}$-module X, we set $X^* := \mathrm{Hom}_{\mathfrak{A}}(X, \mathfrak{A})$ and $X^\vee := \mathrm{Hom}_{\mathfrak{A}}(X, A/\mathfrak{A})$ and regard both as $\mathfrak{A}$-modules in the natural way.

In addition, for an $\mathfrak{A}$-lattice X and an idempotent e of A we set

$$X^e := \{x \in X : e \cdot x = x \text{ in } F \otimes_R X\}.$$

We remark that $X^e = e \cdot X$ if e belongs to $\mathfrak{A}$. The next result concerns the behaviour of certain (integral) idempotent components of dual modules.

Proposition 2.5.2. Let e be an idempotent in A and X be a finitely generated $\mathfrak{A}$-module in Ae.

(i) If X is R-torsion free, then X is reflexive, i.e. $X^{**} = X$.

(ii) $(Xe)^* \cong (X^*)^e$ and if X is R-torsion free, then $X^*e = (X^e)^*$.

(iii) If X is a finite $\mathfrak{A}$-module, then one has $(X^\vee)^\vee = X$.

Proof. Claim (i) follows directly from Bass' result that, if $\mathfrak{A}$ is Gorenstein, then every finitely generated R-torsion-free $\mathfrak{A}$-module is reflexive (cf. [1, Th. 6.2]).

For claim (ii), define $\pi : X \to Xe$ by $x \mapsto xe$. We claim that the assignment $\theta \mapsto \theta \circ \pi$ gives an isomorphism $(Xe)^* \xrightarrow{\sim} (X^*)^e$. To show this, first we note that the assignment is well-defined: for any $\theta \in (Xe)^*$, if we denote θ_F for its induced morphism on $F \otimes (Xe)$, then one has $e(\theta \cdot \pi(x)) = (\theta_F \circ \pi)(xe) = \theta \circ \pi(x)$. Moreover, this assignment is invertible : suppose we are given $\theta \in (X^*)^e$, i.e. θ satisfies $\theta(x) = e\theta(x)$. Then for $x \in X$, $\theta_F(ex) = e\theta_F(x) = \theta(x) \in \mathfrak{A}$

and therefore the restriction of θ_F on Xe belongs to $(Xe)^*$. If X is R-torsion free, then so is Xe. By (i), X and Xe are both reflexive. Therefore, $(Xe)^* \cong (X^*)^e$ implies that $Xe = ((X^*)^e)^*$. The result follows by replacing X with X^*.

To prove claim (iii), we let d be a sufficiently large integer so that there exists a surjection $\mathfrak{A}^d \to X$ and denote its kernel by K. For any finite module X, we have $\mathrm{Hom}_{\mathfrak{A}}(X, \mathfrak{A}) = 0$ as $\mathfrak{A}$ is R-torsion free. Also, by applying the functor $\mathrm{Hom}_{\mathfrak{A}}(X, -)$ to the short exact sequence $0 \to \mathfrak{A} \to A \to A/\mathfrak{A} \to 0$, we have $\mathrm{Ext}^1_{\mathfrak{A}}(X, \mathfrak{A}) = X^\vee$. Since $\mathfrak{A}^d$ is free over $\mathfrak{A}$, we have $\mathrm{Ext}^1_{\mathfrak{A}}(\mathfrak{A}^d, \mathfrak{A}) = 0$. Therefore, if we apply the functor $\mathrm{Hom}_{\mathfrak{A}}(-, \mathfrak{A})$ to the short exact sequence $0 \to K \to \mathfrak{A}^d \to X \to 0$, we obtain another short exact sequence $0 \to (\mathfrak{A}^d)^* \to K^* \to X^\vee \to 0$. Now by applying the functor $\mathrm{Hom}_{\mathfrak{A}}(-, \mathfrak{A})$ to this new sequence and repeating the above argument, we obtain a short exact sequence $0 \to (K^*)^* \to ((\mathfrak{A}^d)^*)^* \to (X^\vee)^\vee \to 0$. Since $\mathfrak{A}^d$, and hence K, are finitely generated and R-torsion free, we have that $((\mathfrak{A}^d)^*)^* = \mathfrak{A}^d$ and $(K^*)^* = K$ by (i). Hence, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (K^*)^* & \longrightarrow & ((\mathfrak{A}^d)^*)^* & \longrightarrow & (X^\vee)^\vee & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & \mathfrak{A}^d & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

implies that $(X^\vee)^\vee = X$, as claimed. $\square$

For any $\mathfrak{A}$-module X we write X_{tor} for the submodule comprising all R-torsion elements. The following lemma offers us an equivalent description of certain idempotent components of a quotient of lattices in terms of the torsion elements.

Lemma 2.5.3. Let X be an $\mathfrak{A}$-lattice and e be an idempotent of A. If C is a full sublattice of X^e, then one has $(X/C)_{\mathrm{tor}} = X^e/C$.

Proof. As C is a full sub-lattice of X^e, one has $F \otimes_R C = F \otimes_R X^e$ and this implies $X^e/C \subset (X/C)_{\mathrm{tor}}$. Conversely, since X is an $\mathfrak{A}$-lattice (and therefore R-torsion-free), any element

$x \in X$ such that $x(1-e) \neq 0$ is not an R-torsion in X/C. This implies $(X/C)_{\mathrm{tor}} \subset X^e/C$. $\square$

2.6 Algebra of exterior power biduals

In this section, we review the construction of the exterior power biduals defined in [16] and record several useful algebraic results that will be used in the rest of this thesis. At the outset, we let X be a finitely generated $\mathfrak{A}$-module and recall that we write $(-)^* := \mathrm{Hom}_{\mathfrak{A}}(-, \mathfrak{A})$ to be the linear dual functor.

Definition 2.6.1. For any non-negative integer a, we define the 'a-th exterior power bidual' of X to be the $\mathfrak{A}$-module by setting

$$\bigcap\nolimits_{\mathfrak{A}}^{a} X := \Big(\bigwedge\nolimits_{\mathfrak{A}}^{a}(X^*)\Big)^*.$$

Remark 2.6.2. The A-module $F \otimes_R X$ is both finitely generated and projective and so there is a natural identification of $F \otimes_R \bigwedge_{\mathfrak{A}}^r X$ with $\bigcap_A^r (F \otimes_R X) = \mathrm{Hom}_{\mathfrak{A}}(\bigwedge_{\mathfrak{A}}^r (X^*), A)$. Consequently, the map $a \mapsto (\Phi \mapsto \Phi(a))$ induces an identification

$$\Big\{a \in F \otimes_R \bigwedge\nolimits_{\mathfrak{A}}^{r} X : \Phi(a) \in \mathfrak{A} \text{ for all } \Phi \in \bigwedge\nolimits_{\mathfrak{A}}^{r}(X^*)\Big\} \xrightarrow{\sim} \bigcap\nolimits_{\mathfrak{A}}^{r} X.$$

In this way, we can regard $\bigcap_{\mathfrak{A}}^r X$ as an $\mathfrak{A}$-submodule of $F \otimes_R \bigwedge_{\mathfrak{A}}^r X$.

The following lemma is important when we study the exterior power biduals in the context of Iwasawa theory.

Lemma 2.6.3. Let p be a prime, Γ be a finite abelian group and M be a torsion-free finitely generated $\mathbb{Z}_p[\Gamma]$-module. Then there exists a natural isomorphism of $\mathbb{Z}_p[\Gamma]$-modules

$$\bigcap\nolimits_{\mathbb{Z}_p[\Gamma]}^{r} M \cong \varprojlim_n \bigcap\nolimits_{\mathbb{Z}/p^n[\Gamma]}^{r} (M/p^n)$$

where the limit is taken with respect to the maps $\bigcap^r_{\mathbb{Z}/p^n[\Gamma]}(M/p^n) \to \bigcap^r_{\mathbb{Z}/p^{n-1}[\Gamma]}(M/p^{n-1})$ that are induced by the natural projections $M/p^n \to M/p^{n-1}$.

Proof. For any module N over a commutative ring R, the functor $\mathrm{Hom}_R(N, \cdot)$ commutes with the inverse limit (see, for example, [50, p. 171]).

In particular, the natural map

$$\bigcap\nolimits^r_{\mathbb{Z}_p[\Gamma]} M := \mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(\bigwedge\nolimits^r_{\mathbb{Z}_p[\Gamma]}(M^*), \mathbb{Z}_p[\Gamma]) \to \varprojlim_n \mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(\bigwedge\nolimits^r_{\mathbb{Z}_p[\Gamma]}(M^*), \mathbb{Z}/p^n[\Gamma])$$

is bijective, where we set $M^* := \mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(M, \mathbb{Z}_p[\Gamma])$ and the limit is taken with respect to the natural transition maps. It therefore suffices to show that, for each n, there is a natural identification

$$\mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(\bigwedge\nolimits^r_{\mathbb{Z}_p[\Gamma]}(M^*), \mathbb{Z}/p^n[\Gamma]) \cong \bigcap\nolimits^r_{\mathbb{Z}/p^n[\Gamma]} M/p^n.$$

But, since M is both finitely generated and torsion-free, for each n there is a natural identification of modules $M^*/p^n \cong (M/p^n)^\vee := \mathrm{Hom}_{\mathbb{Z}/p^n[\Gamma]}(M/p^n, \mathbb{Z}/p^n[\Gamma])$ and hence also an induced identification

$$\begin{aligned}
\bigcap\nolimits^r_{\mathbb{Z}/p^n[\Gamma]} M/p^n &:= \mathrm{Hom}_{\mathbb{Z}/p^n[\Gamma]}(\bigwedge\nolimits^r_{\mathbb{Z}/p^n[\Gamma]}(M/p^n)^\vee, \mathbb{Z}/p^n[\Gamma]) \\
&= \mathrm{Hom}_{\mathbb{Z}/p^n[\Gamma]}(\bigwedge\nolimits^r_{\mathbb{Z}/p^n[\Gamma]}(M^*/p^n), \mathbb{Z}/p^n[\Gamma]) \\
&= \mathrm{Hom}_{\mathbb{Z}_p[\Gamma]}(\bigwedge\nolimits^r_{\mathbb{Z}_p[\Gamma]}(M^*), \mathbb{Z}/p^n[\Gamma]),
\end{aligned}$$

as required.

□

Chapter 3

Higher Special Elements

This chapter presents the joint work with the author's research supervisor David Burns and Takamichi Sano.

In this chapter, we develop the theory of higher special elements of strictly admissible cohomology complexes. In particular, we shall, firstly, prove that these special elements generate a canonical ideal that is simultaneously contained in the higher Fitting ideal of a cohomology group and the annihilator of certain subquotients of it. Building upon this result, we determine completely the structure of the exterior bidual of the cohomology group modulo the group generated by the higher special elements in terms of this canonical ideal.

3.1 The key definitions

We will adopt the notation from §2.3. In particular, we have fixed a Dedekind domain R of characteristic 0 with field of fractions F and a commutative R-order $\mathfrak{A}$ that spans a separable F-algebra $A := F \otimes_R \mathfrak{A}$. In this section, we further fix a strictly admissible complex of $\mathfrak{A}$-modules C and, for some extension field E of F, an isomorphism of A_E-modules of the form

$$\lambda : E \otimes_R H^1(C) \xrightarrow{\sim} E \otimes_R H^2(C).$$

3.1.1 Characteristic elements

The given isomorphism λ induces a composite isomorphism of A_E-modules

$$\begin{aligned} \vartheta_\lambda : \mathrm{Det}_{A_E}(E \otimes_R C) \cong & \mathrm{Det}_{A_E}(E \otimes_R H^1(C))^{-1} \otimes \mathrm{Det}_{A_E}(E \otimes_R H^2(C)) \\ \cong & \mathrm{Det}_{A_E}(E \otimes_R H^2(C))^{-1} \otimes \mathrm{Det}_{A_E}(E \otimes_R H^2(C)) \\ \cong & (A_E, 0), \end{aligned} \tag{3.1}$$

where the first isomorphism is the canonical 'passage to cohomology' isomorphism, the second is $\mathrm{Det}_{A_E}(\lambda)^{-1} \otimes \mathrm{id}$ and the third is induced by the standard evaluation pairing on $\mathrm{Det}_{A_E}(E \otimes_R H^2(C))$.

Definition 3.1.1. An element $\mathcal{L} = \mathcal{L}_{(C,\lambda)}$ of $A_E^\times$ is said to be a 'characteristic element' for the pair (C, λ) if it satisfies

$$\vartheta_\lambda(\mathrm{Det}_{\mathfrak{A}}(C)) = (\mathfrak{A} \cdot \mathcal{L}, 0). \tag{3.2}$$

Remark 3.1.2. There exists an element $\mathcal{L}$ of $A_E^\times$ with the property (3.2) if and only if the Euler characteristic of C in $K_0(\mathfrak{A})$ vanishes. The condition (ad_2) implies that this condition is automatically satisfied if $K_0(\mathfrak{A})$ is torsion-free as is the case, for example, if R is local (see [37]).

For each non-negative integer a, we define $e_{C,a}$ to be the sum of all primitive idempotents e of A with the property that the (free) Ae-module $e(F \cdot H^2(C))$ has rank a and then define

$$e_{C,(a)} := \sum_{a' \geq a} e_{C,a'}.$$

We note that $e_{C,a}$ and $e_{C,(a)}$ are both idempotents of A.

3.1.2 Higher special elements

We are now ready to define the higher special elements associated with a strictly admissible complex.

Definition 3.1.3. *Let $\mathcal{L}$ be a characteristic element for the pair (C, λ). Then, for any ordered subset $\mathcal{X}$ of $H^2(C)_{\mathrm{tf}}$ of cardinality a, the* higher special element *associated to the data $(C, \lambda, \mathcal{L}, \mathcal{X})$ is the unique element $\eta = \eta_{(C,\lambda,\mathcal{L},\mathcal{X})}$ of $\bigwedge_{A_E}^a H^1(C)_E$ that satisfies*

$$(\bigwedge\nolimits_{A_E}^a \lambda)(\eta) = e_{C,a} \cdot \mathcal{L}^{-1} \cdot \wedge_{x\in\mathcal{X}} x. \tag{3.3}$$

Although the higher special elements seem to have coefficients in E by their definitions, the following lemma suggests that these special elements indeed have coefficients over the smaller field F.

Lemma 3.1.4. For any data $(C, \lambda, \mathcal{L}, \mathcal{X})$ as in Definition 3.1.3 the element $\eta_{(C,\lambda,\mathcal{L},\mathcal{X})}$ belongs to $e_{C,|\mathcal{X}|}(F \cdot \bigwedge_{\mathfrak{A}}^{|\mathcal{X}|} H^1(C))$.

Proof. Setting $\eta := \eta_{(C,\lambda,\mathcal{L},\mathcal{X})}$, $a := |\mathcal{X}|$ and $e_a := e_{C,a}$, it is enough to prove for each primitive idempotent e of A that the element $e(\eta)$ belongs to $ee_a(\bigwedge_A^a H^1(C^\bullet))$.

If $ee_a = 0$, then this containment is clear since the defining equality (3.3) implies that $e(\eta) = 0$.

If $ee_a \neq 0$, then the rank over Ae of $e(F \cdot H^2(C))$, and hence also of $e(F \cdot H^1(C))$, is a and so

$$e \cdot \mathrm{Det}_{A_E}(E \otimes_R H^i(C)) = (\bigwedge\nolimits_{A_E}^a e(E \otimes_R H^i(C)), a)$$

for both $i = 1$ and $i = 2$. Given this, our choice of $\mathcal{L}$ implies that

$$(\bigwedge\nolimits_{A_E}^{a}\lambda)(e_a(F\cdot\bigwedge\nolimits_{\mathfrak{A}}^{a}H^1(C))) = e_a\cdot\mathcal{L}^{-1}\cdot(F\cdot\bigwedge\nolimits_{\mathfrak{A}}^{a}H^2(C))$$

and so the required containment is true since $\wedge_{x\in\mathcal{X}}x$ belongs to $\bigwedge_{\mathfrak{A}}^{a}H^2(C)$. $\square$

3.1.3 Separability

The theory of special elements has an especially rich structure when one considers subsets $\mathcal{X}$ of $H^2(C)_{\rm tf}$ with the following property.

Definition 3.1.5. *A finite subset $\mathcal{X}$ of an $\mathfrak{A}$-module X is said to be* separable *if the $\mathfrak{A}$-module $\langle\mathcal{X}\rangle$ that it generates is both free of rank $|\mathcal{X}|$ and a direct summand of X.*

Example 3.1.6. Assume that R is local, with maximal ideal $\mathfrak{m}$, and that $\mathfrak{A} = R[G]$ for a finite abelian group G. Write $G = P \times H$ with P the Sylow p-subgroup. Then, by [10, Proposition 2.1], one knows that a finite subset $\mathcal{X}$ of an $\mathfrak{A}$-lattice X is separable if and only if the images in $H^0(P, X)/\mathfrak{m}$ of the elements $h\cdot\sum_{g\in P}g(x)$ for $h\in H$ and $x\in X$ are linearly independent over $R/\mathfrak{m}$. In particular, if $G = P$, then a singleton subset $\{x\}$ of an $R[G]$-lattice X is separable if and only if the element $\sum_{g\in G}g(x)$ is not contained in $\mathfrak{m}\cdot H^0(G, X)$.

3.2 Higher Fitting ideals and annihilators

In this section, we study the integral properties of higher special elements by establishing that a canonical ideal defined by these elements is contained in a certain higher Fitting ideal of $H^2(C)$.

3.2.1 Statement of the main results

At the outset, we fix data $(C, \lambda, \mathcal{L}, \mathcal{X})$ as in Definition 3.1.3 and denote the associated special element by $\eta = \eta_{(C,\lambda,\mathcal{L},\mathcal{X})}$. For a finitely generated $\mathfrak{A}$-module X, we set $X^* := \mathrm{Hom}_{\mathfrak{A}}(X, \mathfrak{A})$. We note that Lemma 3.1.4 implies the set

$$I(\eta) := \{\Phi(\eta) : \Phi \in \wedge_{\mathfrak{A}}^{|\mathcal{X}|} H^1(C)^*\} \tag{3.4}$$

is an $\mathfrak{A}$-submodule of A. The main results in this section are the following.

Theorem 3.2.1. Fix data $(C, \lambda, \mathcal{L}, \mathcal{X})$ as in Definition 3.1.3 and set $\eta := \eta_{(C,\lambda,\mathcal{L},\mathcal{X})}$, $a := |\mathcal{X}|$ and $\mathfrak{A}' := \mathfrak{A} e_{C,(a)}$. Then for any element y of the product ideal

$$\mathrm{Ann}_{\mathfrak{A}}(\mathrm{Ext}^2_{\mathfrak{A}}(H^2(C), \mathfrak{A})) \cdot \mathrm{Ann}_{\mathfrak{A}}(\mathrm{Ext}^2_{\mathfrak{A}'}(\mathfrak{A}' \otimes_{\mathfrak{A}} H^2(C), \mathfrak{A}'))$$

the following claims are valid.

(i) If $H^2(C)'$ is any subquotient of $H^2(C)$ such that for each primitive idempotent of $Ae_{(a)}$, the (free) Ae-module $e(F \cdot H^2(C)')$ has rank at least a, then for any element x of $\mathfrak{A} \cap \mathfrak{A}'$ one has

$$xy^a \cdot I(\eta) \subseteq \mathrm{Fit}^a_{\mathfrak{A}}(H^2(C)) \cap \mathrm{Ann}_{\mathfrak{A}}(H^2(C)'_{\mathrm{tor}}).$$

(ii) If $\mathcal{X}$ is separable, then $e_{C,(a)} = 1$ and both

$$y^a \cdot I(\eta) \subseteq \mathrm{Fit}^a_{\mathfrak{A}}(H^2(C)) \text{ and } \mathrm{Fit}^a_{\mathfrak{A}}(H^2(C)) \subseteq I(\eta).$$

Remark 3.2.2. In certain cases, the elements x and y can be either omitted from, or made explicit, in the statement of Theorem 3.2.1. For example, if $e = e_{C,(a)} \in \mathfrak{A}$, then the element x can be taken to be e and so (since $\eta = e \cdot \eta$) can be omitted from the statement. Regarding the choice of y, note for example, that if $\mathfrak{B}$ is an R-order such that $\mathrm{Ext}^1_{\mathfrak{B}}(N,\mathfrak{B})$ vanishes for all finitely generated torsion-free $\mathfrak{B}$-modules N (see Example 2.3.7(ii)), then $\mathrm{Ext}^2_{\mathfrak{B}}(X',\mathfrak{B})$ vanishes for all finitely generated $\mathfrak{B}$-modules X'.

Remark 3.2.3. Under the conditions discussed in Remark 3.2.2 (for example when $\mathcal{X}$ is separable and $\mathfrak{A}$ is Gorenstein), Theorem 3.2.1 implies that $I(\eta_{(C,\lambda,\mathcal{L},\mathcal{X})}) = \mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))$ with $a = |\mathcal{X}|$ and so the special element completely determines the higher Fitting ideal.

Remark 3.2.4. If $\mathfrak{A} = R[\Gamma]$ for a local ring R and a finite abelian group Γ, then the containment in Theorem 3.2.1, in particular, imply $xy^a \cdot I(\eta) \subseteq \mathfrak{A}$. This further implies families of congruence relations between the different components of elements of the form $(\wedge_{i=1}^{i=a}\varphi_i)(\eta)$. To see this, we define for each ρ in $\Gamma^* := \mathrm{Hom}(\Gamma, F^{c,\times})$ an idempotent $e_\rho := |\Gamma|^{-1}\sum_{\gamma\in\Gamma}\rho(\gamma^{-1})\gamma$ in $F^c[\Gamma]$ and then for each x in $F^c[\Gamma]$ we define elements x_ρ in F^c via the equality

$$x = \sum_{\rho\in\Gamma^*} x_\rho e_\rho = |\Gamma|^{-1}\sum_{\gamma\in\Gamma}\gamma\sum_{\rho\in\Gamma^*}\rho(\gamma^{-1})x_\rho.$$

Then, writing F_ρ for the field generated over F by $\{\rho(\gamma) : \gamma \in \Gamma\}$ and $\mathcal{O}_\rho$ for its valuation ring, an element x belongs to $R[\Gamma]$ if and only if one has $x_\rho \in \mathcal{O}_\rho$ for all ρ in Γ^*, $\omega(x_\rho) = x_{\omega\circ\rho}$ for all $\rho \in \Gamma^*$ and $\omega \in G_{F_\rho/F}$ and $\sum_{\rho\in\Gamma^*}\rho(\gamma^{-1})x_\rho \equiv 0$ (modulo $|\Gamma|\cdot R$) for all $\gamma \in \Gamma$.

Here we record an easy consequence of Theorem 3.2.1(i).

Theorem 3.2.5. Fix data $\mathfrak{A}, C, \lambda$ and $\mathcal{L}$ as in Theorem 3.2.1. Then for any non-negative integer a and any elements x and y of $\mathfrak{A}$ as in Theorem 3.2.1(i) the following claims are valid.

(i) For any ordered subset $\mathcal{X}$ of $H^2(C)_{\mathrm{tf}}$ such that $|\mathcal{X}| = a$, the higher special element $\eta_{\mathcal{X}}$ associated with the data $(C,\lambda,\mathcal{L},\mathcal{X})$ satisfies $xy^a \cdot \eta_{\mathcal{X}} \in \bigcap^a_{\mathfrak{A}} H^1(C)$.

(ii) There is an inclusion

$$xy^a \cdot e_{C,a} \cdot \mathcal{L}^{-1} \cdot (\bigwedge\nolimits_{\mathfrak{A}}^{a} H^2(C))_{\mathrm{tf}} \subseteq (\bigwedge\nolimits_{A_E}^{a} \lambda)(\bigcap\nolimits_{\mathfrak{A}}^{a} H^1(C)).$$

Proof. Since $\mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))$ is contained in $\mathfrak{A}$, the containment in claim (i) follows immediately from the containment in Theorem 3.2.1(i).

Next we note that the $\mathfrak{A}$-module $(\bigwedge^a_{\mathfrak{A}} H^2(C))_{\mathrm{tf}}$ is spanned by elements of the form $\wedge_{x\in\mathcal{X}} x$ where $\mathcal{X}$ runs over (ordered) subsets of $H^2(C)$ of cardinality a.

This implies that the $\mathfrak{A}$-module $e_{C,a} \cdot \mathcal{L}^{-1} \cdot (\bigwedge^a_{\mathfrak{A}} H^2(C))_{\mathrm{tf}}$ is generated by elements of the form $(\bigwedge^a_{A_E} \lambda)(\eta_{\mathcal{X}})$ and so claim (ii) follows directly from claim (i). $\square$

3.2.2 Proof of Theorem 3.2.1 (i)

We first prove an important reduction result.

Proposition 3.2.6. It is enough to prove Theorem 3.2.1 (i) in the case that $H^2(C)'$ is a quotient of $H^2(C)$ and the image of $\mathcal{X}$ in $e_{(a)}(F \cdot H^2(C)')$ generates a free $\mathfrak{A}e_{(a)}$-module of rank a.

Proof. The first assertion is clear since if $H^2(C)'$ is a submodule of a quotient Q of $H^2(C)$, then Q must have rank at least a at each simple component of $A_{(a)}$ and $H^2(C)'_{\mathrm{tor}}$ is a submodule of Q_{tor} so that $\mathrm{Ann}_{\mathfrak{A}}(Q_{\mathrm{tor}}) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^2(C)'_{\mathrm{tor}})$.

To prove the second assertion we assume that $Q := H^2(C)'$ is a quotient of $H^2(C)$, label the elements of $\mathcal{X}$ as $\{x_i\}_{1\le i\le a}$ and for each index i write $\overline{x_i}$ for the image of x_i in Q.

We can also fix a subset $\mathcal{Y} := \{y_j\}_{1\le j\le a}$ of Q that spans a free Ae_a-module of rank a. Then Lemma 3.2.7 below implies that for any large enough integer N the set $\mathcal{Y}_N := \{\overline{x_j} + p^N \cdot y_j\}_{1\le j\le a}$ spans a free $Ae_{(a)}$-module of rank a. We fix such an N and write η_N for the unique element that satisfies

$$(\bigwedge\nolimits_{A_E}^{a}\lambda)(\eta_N) = e_a \cdot \mathcal{L}^{-1} \cdot \wedge_{y\in\mathcal{Y}_N} y.$$

By explicitly comparing the terms $\wedge_{y\in\mathcal{Y}_N} y$ and $\wedge_{x\in\mathcal{X}} x$ one finds that

$$\eta_N \equiv \eta \text{ modulo } p^N(\bigwedge\nolimits_{A_E}^{a}\lambda)^{-1}(\mathcal{L}^{-1}\cdot\Xi)$$

with Ξ the $\mathfrak{A}$-submodule of Q generated by $\mathcal{X}\cup\mathcal{Y}$, and so it enough for us to prove that the $\mathfrak{A}$-lattice

$$I(\Xi) := \{(\wedge_{i=1}^{i=|\mathcal{X}|}\varphi_i)(\bigwedge\nolimits_{A_E}^{a}\lambda)^{-1}(\mathcal{L}^{-1}\cdot\xi) : \xi \in \Xi \text{ and } (\varphi_i)_i \in \mathrm{Hom}_{\mathfrak{A}}(H^1(C),\mathfrak{A})^{|\mathcal{X}|}\}$$

is such that for any large enough choice of N one has

$$p^N \cdot I(\Xi) \subseteq \mathrm{Fit}^a_{\mathfrak{A}}(H^2(C)) \cap \mathrm{Ann}_{\mathfrak{A}}(Q_{\mathrm{tor}}). \tag{3.5}$$

To show this we note that the definition of e_a combines with our choice of $\mathcal{Y}$ to imply that the natural projection and inclusion maps

$$e_a(F\cdot(\bigwedge\nolimits_{\mathfrak{A}}^{a}H^2(C))) \to e_a(F\cdot(\bigwedge\nolimits_{\mathfrak{A}}^{a}Q)) \quad\text{and}\quad e_a(F\cdot(\bigwedge\nolimits_{\mathfrak{A}}^{a}\Xi)) \to e_a(F\cdot(\bigwedge\nolimits_{\mathfrak{A}}^{a}Q)$$

are both bijective. This implies that $(\bigwedge^a_{A_E}\lambda)^{-1}(\mathcal{L}^{-1}\cdot\Xi)$ is a full $\mathfrak{A}$-sublattice of the space $e_a(F\cdot(\bigwedge^a_{\mathfrak{A}}H^1(C)))$ and hence that $I(\Xi)$ is a full sublattice of e_aA.

Since $\mathrm{Ann}_{\mathfrak{A}}(Q_{\mathrm{tor}})$ has finite index in $\mathfrak{A}$, to prove that (3.5) is valid for any large enough N, it is thus enough to note that $\mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))\cap e_aA$ has finite index in $e_a\mathfrak{A}$. This is true because $e_a(F \cdot H^2(C))$ is a free A-module of rank a and hence that $e_a(F \cdot \mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))) = \mathrm{Fit}^a_{Ae_a}(e_a(F \cdot H^2(C))) = Ae_a$. $\square$

In the following result we use the set $\mathcal{Y}_N$ defined above.

Lemma 3.2.7. For any large enough natural number N the $Ae_{(a)}$-module Y_N spanned by $\mathcal{Y}_N$ is free of rank a.

Proof. The algebra $Ae_{(a)}$ decomposes as a finite product of fields (of characteristic zero) and, after replacing $Ae_{(a)}$ by any such field E, it is enough to prove that for any large enough natural number N the E-module $E \otimes_{\mathfrak{A}} Y_N$ is free of rank a.

We write π for the natural projection $Q \to E \otimes_{\mathfrak{A}} Y_N$. Then the set $\pi(\mathcal{Y})$ gives an E-basis of the space $E \otimes_{\mathfrak{A}} Q = E \otimes_{\mathfrak{A}} H^2(C)$ and we write M_N and M for the transition matrices from $\pi(\mathcal{Y})$ to the sets $\pi(\mathcal{Y}_N)$ and $\pi(\mathcal{X})$. It is then enough to prove that for any large enough N the matrix M_N is invertible.

However, since $M_N = M + p^N \cdot \mathrm{Id}_a$, this is true since for any large enough choice of N the integer $-p^N$ cannot be an eigenvalue of M (over any algebraic closure of E). $\square$

It is enough to prove Theorem 3.2.1 (i) after replacing $\mathfrak{A}$ by its localisation $\mathfrak{A}_{\mathfrak{p}}$ at each prime ideal $\mathfrak{p}$ of R and then the semi-local ring $\mathfrak{A}_{\mathfrak{p}}$ by each of its local components. In the sequel we shall therefore assume that $\mathfrak{A}$ is local and hence that every finitely generated projective $\mathfrak{A}$-module is free.

In addition, we always assume $H^2(C)'$ and $\mathcal{X}$ are chosen as in Proposition 3.2.6. We also write e_a and $e_{(a)}$ in place of $e_{C,a}$ and $e_{C,(a)}$ and then set $\mathfrak{A}_a := \mathfrak{A}e_a$, $\mathfrak{A}_{(a)} := \mathfrak{A}e_{(a)}$, $A_a := Ae_a$ and $A_{(a)} := Ae_{(a)}$.

We consider the object $C_{(a)} := \mathfrak{A}_{(a)} \otimes^{\mathbb{L}}_{\mathfrak{A}} C$ of $D^{\mathrm{p}}(\mathfrak{A}_{(a)})$.

We note that the definition of $e_{(a)}$ ensures $F \cdot H^2(C_{(a)})$ contains a free $A_{(a)}$-module of rank a and then Remark 2.3.2 implies the same is true of the $A_{(a)}$-module $F \cdot H^1(C_{(a)})$. We can thus fix a subset $\mathcal{X}' = \{x'_j\}_{1\le j\le a}$ of $H^1(C_{(a)})$ that is linearly independent over $A_{(a)}$.

In particular, since the definition of e_a ensures that the sets $e_a\mathcal{X}'$ and $e_a\mathcal{X}$ are respectively bases of the $A_{a,E}$-modules $e_a(E\cdot H^1(C_{(a)}))$ and $e_a(E\cdot H^2(C_{(a)}))$, we can define $M(\lambda)$ to be the matrix in $\mathrm{GL}_a(A_{a,E})$ that represents $e_a\lambda$ with respect to these bases.

The next key step is to prove the following result concerning this matrix.

Lemma 3.2.8. Set $\mathfrak{A}' := \mathfrak{A}_{(a)}$ and $C' := C_{(a)}$. Then there exists a natural homomorphism of $\mathfrak{A}'$-modules

$$\kappa : \mathrm{Ext}^1_{\mathfrak{A}'}(H^1(C'),\mathfrak{A}') \to \mathrm{Ext}^3_{\mathfrak{A}'}(H^2(C'),\mathfrak{A}')$$

for which one has

$$\mathrm{Ann}_{\mathfrak{A}'}(\mathrm{Ext}^2_{\mathfrak{A}'}(H^2(C'),\mathfrak{A}'))^a \cdot \mathrm{Fit}^0_{\mathfrak{A}'}(\ker(\kappa)) \cdot (\det(M(\lambda))\mathcal{L})^{-1} \subseteq \mathrm{Fit}^0_{\mathfrak{A}'}(H^2(C')/X).$$

Proof. Since the $\mathfrak{A}'$-modules X and X' that are generated by $\mathcal{X}$ and $\mathcal{X}'$ are both free of rank a, the inclusions $\iota^2 : X \subseteq H^2(C')$ and $\iota^1 : X' \subseteq H^1(C')$ give rise to an exact triangle in $D^{\mathrm{p}}(\mathfrak{A}')$ of the form

$$X[-2] \oplus X'[-1] \xrightarrow{\iota} C' \to D \to X[-1] \oplus X'[0] \tag{3.6}$$

in which $H^1(\iota) = \iota^1$, $H^2(\iota) = \iota^2$ and the complex D is acyclic outside degrees one and two and has cohomology groups in these degrees that respectively identify with the quotients $H^1(C')/X'$ and $H^2(C')/X$.

Since λ induces an isomorphism of A_E-modules between $e_a(F\cdot X') = e_a(F\cdot H^1(C))$ and

$e_a(F \cdot H^2(C)) = e_a(F \cdot X)$ we may fix a commutative diagram of A_E-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E \cdot X' & \xrightarrow{\subseteq} & E \cdot H^1(C') & \longrightarrow & E \cdot H^1(D) & \longrightarrow & 0 \\ & & {\scriptstyle \lambda_1}\big\downarrow & & {\scriptstyle \lambda_2}\big\downarrow & & {\scriptstyle \lambda_3}\big\downarrow & & \\ 0 & \longrightarrow & E \cdot X & \xrightarrow{\subseteq} & E \cdot H^2(C') & \longrightarrow & E \cdot H^2(D) & \longrightarrow & 0 \end{array} \tag{3.7}$$

where the maps λ_1, λ_2 and λ_3 are bijective and satisfy

$$e_a\lambda_1 = e_a\lambda_2 = e_a\lambda. \tag{3.8}$$

This commutative diagram combines with the triangle (3.6) to give an equality of lattices

$$\vartheta_{\lambda_3}(\mathrm{Det}_{\mathfrak{A}'}(D)) = \vartheta_{\lambda_1}(\mathrm{Det}_{\mathfrak{A}'}(X[-2] \oplus X'[-1]))^{-1} \cdot \vartheta_{\lambda_2}(\mathrm{Det}_{\mathfrak{A}'}(C')).$$

In particular, upon multiplying this equality by e_a, and taking account of both of the equalities (3.2) and (3.8), one deduces that

$$\begin{aligned} e_a\vartheta_{\lambda_3}(\mathrm{Det}_{\mathfrak{A}'}(D)) =& e_a\vartheta_{\lambda_1}(\mathrm{Det}_{\mathfrak{A}'}(X[-2] \oplus X'[-1]))^{-1} \cdot e_a\vartheta_{\lambda_2}(\mathrm{Det}_{\mathfrak{A}'}(C')) \\ =& \det(M(\lambda)) \cdot \vartheta_{e_a\lambda_2}(e_a \cdot \mathrm{Det}_{\mathfrak{A}}(C)) \\ =& \det(M(\lambda)) \cdot e_a\vartheta_{\lambda}(\mathrm{Det}_{\mathfrak{A}}(C)) \\ =& \mathfrak{A} \cdot \det(M(\lambda))\mathcal{L} \end{aligned} \tag{3.9}$$

where $M(\lambda)$ is the matrix defined above.

To investigate the first expression in (3.9) we note that Remark 2.3.4 allows us to fix a representative of C of the form $P^1 \to P^2$, where P^1 and P^2 are finitely generated free $\mathfrak{A}$-modules, P^1 is placed in degree one and the differential is δ. In this case the complex C' is represented by the complex $e_{(a)}P^1 \to e_{(a)}P^2$ with differential $e_{(a)}\delta$ and so we can take D to be a complex

$$X' \xrightarrow{\iota' \oplus 0} e_{(a)}P^1 \oplus X \xrightarrow{(e_{(a)}\delta, \iota)} e_{(a)}P^2 \tag{3.10}$$

with X' placed in degree zero.

After applying the result of Proposition 3.2.9(ii) below to this complex we obtain the claimed inclusion as a direct consequence of (3.9). $\square$

Proposition 3.2.9. Let $\mathfrak{B}$ be a commutative local R-algebra that spans an F-algebra B. Let D be a complex of finitely generated free $\mathfrak{B}$-modules of the form

$$D^0 \xrightarrow{\delta^0} D^1 \xrightarrow{\delta^1} D^2$$

where D^0 is placed in degree zero. Assume D is acyclic outside degrees one and two and that there is an isomorphism of B_E-modules of the form $\mu : E \cdot H^1(D) \cong E \cdot H^2(D)$.

Then the following claims are valid.

(i) There exists a natural homomorphism of abelian groups of the form

$$\kappa : \mathrm{Ext}^1_{\mathfrak{B}}(H^1(D),\mathfrak{B}) \to \mathrm{Ext}^3_{\mathfrak{B}}(H^2(D),\mathfrak{B}).$$

(ii) With e denoting the sum of primitive idempotents of B that annihilate $F \cdot H^2(D)$, one has

$$\mathrm{Ann}_{\mathfrak{B}}(\mathrm{Ext}^2_{\mathfrak{B}}(H^2(D),\mathfrak{B}))^{\mathrm{rk}(D^0)} \cdot \mathrm{Fit}^0_{\mathfrak{B}}(\ker(\kappa)) \cdot e \cdot \vartheta_\mu(\mathrm{Det}_{\mathfrak{B}}(D))^{-1} \subseteq \mathrm{Fit}^0_{\mathfrak{B}}(H^2(D)),$$

where $\mathrm{rk}(D^0)$ denotes the rank of the $\mathfrak{B}$-module D^0.

Proof. The differential δ^0 is injective and, since the groups $e(F \cdot H^2(D))$ and hence $e(F \cdot H^1(D))$ vanish, there exists a direct sum decomposition $e(F \cdot D^1) = V_1^1 \bigoplus V_2^1$ so that the maps

$e(F\otimes_R \delta^0)$ and $e(E\otimes_R \delta^1)$ give isomorphisms $e(F\cdot D^0)\cong V_1^1$ and $V_2^1\cong e(F\cdot D^2)$ respectively. We can therefore fix an isomorphism of B_E-modules

$$\iota : E\cdot(D^0\oplus D^2)\to E\cdot D^1 \tag{3.11}$$

which restricts to give the scalar extension of the isomorphism $e(F\cdot D^0)\oplus e(F\cdot D^2)\cong e(F\cdot D^1)=V_1^1\oplus V_2^1$ given by $(e(F\otimes_R \delta^0), e(F\otimes_R \delta^1)^{-1})$.

For any such isomorphism one has an equality

$$e\cdot\vartheta_\mu(\mathrm{Det}_{\mathfrak{B}}(D))=\mathfrak{B}e\cdot\det(M(\iota)) \tag{3.12}$$

where $M(\iota)$ is the matrix of ι with respect to any choice of $\mathfrak{B}$-bases of $D^0\oplus D^2$ and D^1.

To compute the term $\det(M(\iota))$ more explicitly we apply the functor $\mathrm{Hom}_{\mathfrak{B}}(-,\mathfrak{B})$ to the tautological exact sequences

$$\begin{cases} 0\to D^0\xrightarrow{\delta^0} Z^1(D)\to H^1(D)\to 0\\ 0\to Z^1(D)\to D^1\to B^2(D)\to 0\\ 0\to B^2(D)\to D^2\to H^2(D)\to 0\end{cases}$$

In particular, since the groups $\mathrm{Ext}^j_{\mathfrak{B}}(D^i,\mathfrak{B})$ vanish for each $j\geq 1$ and each $i\in\{0,1,2\}$, we obtain in this way exact sequences

$$\mathrm{Hom}_{\mathfrak{B}}(Z^1(D),\mathfrak{B})\xrightarrow{j^0}\mathrm{Hom}_{\mathfrak{B}}(D^0,\mathfrak{B})\xrightarrow{j^1}\mathrm{Ext}^1_{\mathfrak{B}}(H^1(D),\mathfrak{B})\xrightarrow{j^2}\mathrm{Ext}^1_{\mathfrak{B}}(Z^1(D),\mathfrak{B})\to 0, \tag{3.13}$$

and

$$\mathrm{Hom}_{\mathfrak{B}}(D^1,\mathfrak{B})\xrightarrow{k^0}\mathrm{Hom}_{\mathfrak{B}}(Z^1(D),\mathfrak{B})\xrightarrow{k^1}\mathrm{Ext}^1_{\mathfrak{B}}(B^2(D),\mathfrak{B})\to 0$$

and isomorphisms $\mathrm{Ext}^1_{\mathfrak{B}}(Z^1(D),\mathfrak{B}) \cong \mathrm{Ext}^2_{\mathfrak{B}}(B^2(D),\mathfrak{B}) \cong \mathrm{Ext}^3_{\mathfrak{B}}(H^2(D),\mathfrak{B})$.

Taking the composite of the latter isomorphism with the map j^2 in (3.13) we obtain a map κ as in claim (i).

Turning to claim (ii) we set $t := \mathrm{rk}(D^0)$ and choose a $\mathfrak{B}$-basis $\{y_i\}_{1\le i\le t}$ of D^0. Then the $\mathfrak{B}$-module $\mathrm{Hom}_{\mathfrak{B}}(D^0,\mathfrak{B})$ is free with basis $\{y_j^*\}_{1\le j\le t}$ where each y_j^* is the dual of y_j.

For each integer j with $1 \le j \le t$ we choose an element

$$b_j := \sum_{i=1}^{i=t} c_{ij} y_i^* \tag{3.14}$$

in $\ker(j^1)$ and an element ϕ_j of $\mathrm{Hom}_{\mathfrak{B}}(Z^1(D),\mathfrak{B})$ with $j^0(\phi_j) = b_j$. Then for any element z of the group

$$\mathrm{Ann}_{\mathfrak{B}}(\mathrm{Ext}^1_{\mathfrak{B}}(B^2(D),\mathfrak{B})) = \mathrm{Ann}_{\mathfrak{B}}(\mathrm{Ext}^2_{\mathfrak{B}}(H^2(D),\mathfrak{B}))$$

there exists a homomorphism φ_j in $\mathrm{Hom}_{\mathfrak{B}}(D^1,\mathfrak{B})$ with $k^0(\varphi_j) = z\cdot\phi_j$.

We finally define ϕ to be the element of $\mathrm{Hom}_{\mathfrak{B}}(D^1,D^0)$ that sends each element w of D^1 to $\sum_{i=1}^{i=t}\varphi_i(w)\cdot y_i$ and consider the homomorphism $D^1 \to D^0 \oplus D^2$ that is given by the direct sum $\phi\oplus\delta^1$.

Now, by explicitly comparing this map to the isomorphism ι defined in (3.11) one computes that on $e(E\cdot D^0)\oplus e(E\cdot D^2)$ there is an equality of functions

$$e(E\otimes_R(\phi\oplus\delta^1))\circ e(\iota) = (e(E\otimes_R(\phi\circ\delta^0)),\mathrm{id}_{e(E\cdot D^2)}).$$

and for each basis element y_i one has

$$(\phi \circ \delta^0)(y_i) = \sum_{j=1}^{j=t} z(\phi_j \circ \delta^0)(y_i) \cdot y_j = \sum_{j=1}^{j=t} z(b_j)(y_i) \cdot y_j$$
$$= \sum_{j=1}^{j=t} z(\sum_{a=1}^{a=t} c_{aj} y_a^*(y_i)) \cdot y_j = \sum_{j=1}^{j=t} zc_{ij} \cdot y_j.$$

Hence, if we write $M(\phi \oplus \delta^1)$ for the matrix of $\phi \oplus \delta^1$ with respect to any choice of $\mathfrak{B}$-bases of D^1 and D^2 and the fixed basis $\{y_i\}_{1 \le i \le t}$ of D^0, then one has

$$\det(e \cdot M(\phi \oplus \delta^1)) = \det((zc_{ij})_{1 \le i,j \le t}) \cdot \det(e \cdot M(\iota))^{-1}.$$

In addition, for any primitive idempotent e' of B, the matrix $e' \cdot M(\phi \oplus \delta^1)$ is invertible only if $e' = e'e$, and so one has $\det(M(\phi \oplus \delta^1)) = \det(e \cdot M(\phi \oplus \delta^1))$.

Putting everything together we find that (3.12) implies that

$$\begin{aligned} z^t \cdot \det((c_{ij})_{1 \le i,j \le t})e \cdot \vartheta_\mu(\mathrm{Det}_{\mathfrak{B}}(D))^{-1} &= \det((zc_{ij})_{1 \le i,j \le t})e \cdot \vartheta_\mu(\mathrm{Det}_{\mathfrak{B}}(D))^{-1} \\ &= \mathfrak{B}e \cdot \det((zc_{ij})_{1 \le i,j \le t}) \cdot \det(M(\iota))^{-1} \\ &= \mathfrak{B} \cdot \det((zc_{ij})_{1 \le i,j \le t}) \cdot \det(e \cdot M(\iota))^{-1} \\ &= \mathfrak{B} \cdot \det(e \cdot M(\phi \oplus \delta^1)) \\ &= \mathfrak{B} \cdot \det(M(\phi \oplus \delta^1)) \\ &= \mathrm{Fit}^0_{\mathfrak{B}}(\mathrm{cok}(\phi \oplus \delta^1)) \\ &\subseteq \mathrm{Fit}^0_{\mathfrak{B}}(H^2(D)), \end{aligned} \tag{3.15}$$

where the last equality follows directly from the definition of zero-th Fitting ideal and the upper row in the following exact commutative diagram

$$\begin{tikzcd}
D^1 \arrow[r, "\phi\oplus\delta^1"] \arrow[d, equal] & D^0 \oplus D^2 \arrow[r] \arrow[d, "(0,\mathrm{id})"'] & \mathrm{cok}(\phi\oplus\delta^1) \arrow[r] \arrow[d, "\epsilon"'] & 0 \\
D^1 \arrow[r, "\delta^1"] & D^2 \arrow[r] & H^2(D) \arrow[r] & 0,
\end{tikzcd}$$

and the inclusion in (3.15) from (a standard property of Fitting ideals with respect to surjective maps and) the fact that the map ϵ in this diagram is surjective.

Now the exactness of the sequence (3.13) implies that as the elements b_j in (3.14) range over $\ker(j^1)$ the determinants of the matrices $(c_{ij})_{1\leq i,j\leq t}$ range over a set of generators of the ideal $\mathrm{Fit}^0_{\mathfrak{B}}(\ker(j^2)) = \mathrm{Fit}^0_{\mathfrak{B}}(\ker(\kappa))$.

The inclusion (3.15) therefore implies that for any element z of $\mathrm{Ann}_{\mathfrak{B}}(\mathrm{Ext}^2_{\mathfrak{B}}(H^2(D),\mathfrak{B}))$ one has $z^t \cdot \mathrm{Fit}^0_{\mathfrak{B}}(\ker(\kappa)) \cdot \vartheta_\mu(\mathrm{Det}_{\mathfrak{B}}(D))^{-1} \subseteq \mathrm{Fit}^0_{\mathfrak{B}}(H^2(D))$, as required to prove claim (ii).

□

The next result relates the element $\eta_{\mathcal{X}}$ in Theorem 3.2.1 to the terms that occur in Lemma 3.2.8.

Lemma 3.2.10. For each subset $\{\varphi'_j\}_{1\leq j\leq a}$ of $\mathrm{Hom}_{\mathfrak{A}'}(H^1(C'),\mathfrak{A}')$ one has

$$(\wedge_{i=1}^{i=a}\varphi'_i)(\eta_{\mathcal{X}}) \in \mathrm{Fit}^0_{\mathfrak{A}'}(\ker(\kappa)) \cdot (\det(M(\lambda))\mathcal{L})^{-1},$$

where κ is the homomorphism in Lemma 3.2.8.

Proof. To do this we note first that the image of $\eta_{\mathcal{X}}$ under the injective map λ is, by its very definition, equal to

$$\begin{aligned}
e_a \cdot \mathcal{L}^{-1} \cdot \wedge_{i=1}^{i=a} x_i =& \mathcal{L}^{-1} \cdot \wedge_{i=1}^{i=a} e_a x_i \\
=& \mathcal{L}^{-1} \cdot \det(M(\lambda))^{-1} \cdot \lambda(\wedge_{j=1}^{j=a} e_a x'_j) \\
=& (\det(M(\lambda))\mathcal{L})^{-1} \cdot \lambda(\wedge_{j=1}^{j=a} x'_j) \\
=& \lambda((\det(M(\lambda))\mathcal{L})^{-1} \cdot (\wedge_{j=1}^{j=a} x'_j))
\end{aligned}$$

and hence that

$$\eta_{\mathcal{X}} = (\det(M(\lambda))\mathcal{L})^{-1} \cdot \wedge_{j=1}^{j=a} x'_j. \tag{3.16}$$

We next apply the functor $\mathrm{Hom}_{\mathfrak{A}'}(-, \mathfrak{A}')$ to the short exact sequence

$$0 \to X' \xrightarrow{\iota'} H^1(C') \to H^1(D) \to 0$$

to obtain an exact sequence

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{A}'}(H^1(C'), \mathfrak{A}') \xrightarrow{\ell^0} \mathrm{Hom}_{\mathfrak{A}'}(X', \mathfrak{A}') \xrightarrow{\ell^1} \mathrm{Ext}^1_{\mathfrak{A}'}(H^1(D), \mathfrak{A}') \\
\xrightarrow{\ell^2} \mathrm{Ext}^1_{\mathfrak{A}'}(H^1(C'), \mathfrak{A}') \to 0.
\end{aligned} \tag{3.17}$$

Now the $\mathfrak{A}'$-module $\mathrm{Hom}_{\mathfrak{A}'}(X', \mathfrak{A}')$ is free on the basis $\{x'^*_j\}_{1 \le j \le a}$ where each x'^*_j is dual to x'_j. In particular, for the given homomorphisms φ'_j we can write the element $\ell^0(\varphi'_j) = \varphi'_j \circ \iota'$ of $\ker(\ell^1)$ as $\sum_{i=1}^{i=a} c'_{ij} x'^*_i$ with each c'_{ij} in $\mathfrak{A}'$.

The equality (3.16) therefore implies that

$$\begin{aligned}
\det(M(\lambda))\mathcal{L}\cdot(\wedge_{i=1}^{i=a}\varphi_i')(e_a\cdot\eta_{\mathcal{X}}) &= (\wedge_{i=1}^{i=a}\varphi_i')(\wedge_{j=}^{j=a}x_j') \qquad (3.18)\\
&= (\wedge_{i=1}^{i=a}(\varphi_i'\circ\iota))(\wedge_{j=1}^{j=a}x_j')\\
&= \det(((\varphi_i'\circ\iota)(x_j'))_{1\le i,j\le a})\\
&= \det((\sum_{m=1}^{m=a}c'_{mi}x'^{*}_{m})(x_j'))_{1\le i,j\le a})\\
&= \det((c'_{ji})_{1\le i,j\le a})\\
&\in \mathrm{Fit}^0_{\mathfrak{A}_0}(\ker(\ell^2)),
\end{aligned}$$

where the containment follows from the exactness of (3.17).

Finally, we note that, since D is the complex (3.10), one has $Z^1(D)=\ker(e_{(a)}\delta)=H^1(C')$ and the map ℓ^2 coincides with the map j^2 in (3.13). It follows that $\ker(\ell^2)=\ker(j^2)=\ker(\kappa)$, with κ the map in Lemma 3.2.8, and so the claimed result follows directly from (3.18). □

In view of Lemma 3.2.10 it is important to explain the connection between the homomorphism groups $\mathrm{Hom}_{\mathfrak{A}}(H^1(C),\mathfrak{A})$ and $\mathrm{Hom}_{\mathfrak{A}'}(H^1(C'),\mathfrak{A}')$.

Lemma 3.2.11. For each y in $\mathrm{Ann}_{\mathfrak{A}}(\mathrm{Ext}^2_{\mathfrak{A}}(H^2(C),\mathfrak{A}))$ and each φ in $\mathrm{Hom}_{\mathfrak{A}}(H^1(C),\mathfrak{A})$ the product $ye_a\cdot\varphi$ belongs to $\mathrm{Hom}_{\mathfrak{A}'}(H^1(C'),\mathfrak{A}')$.

Proof. The tautological exact sequences

$$\begin{cases} 0\to H^1(C)\to P^1\to B^2(C)\to 0,\\ 0\to B^2(C)\to P^2\to H^2(C)\to 0\end{cases}$$

give rise to an exact sequence

$$\mathrm{Hom}_{\mathfrak{A}}(P^1,\mathfrak{A})\to\mathrm{Hom}_{\mathfrak{A}}(H^1(C),\mathfrak{A})\to\mathrm{Ext}^1_{\mathfrak{A}}(B^2(C),\mathfrak{A})\to 0$$

and an isomorphism $\mathrm{Ext}^1_{\mathfrak{A}}(B^2(C),\mathfrak{A}) \cong \mathrm{Ext}^2_{\mathfrak{A}}(H^2(C),\mathfrak{A})$.

Thus for each y in $\mathrm{Ann}_{\mathfrak{A}}(\mathrm{Ext}^2_{\mathfrak{A}}(H^2(C),\mathfrak{A}))$ and each φ in $\mathrm{Hom}_{\mathfrak{A}}(H^1(C),\mathfrak{A})$ there exists a homomorphism φ_y in $\mathrm{Hom}_{\mathfrak{A}}(P^1,\mathfrak{A})$ which restricts to give $y\cdot\varphi$ on $H^1(C)$.

Since $H^1(C')$ is a submodule of e_aP^1 one therefore has

$$(ye_a\cdot\varphi)(H^1(C')) = \varphi_y(H^1(C')) \subseteq \varphi_y(eP^1) = \varphi_y(P^1)\cdot e \subseteq \mathfrak{A}\cdot e = \mathfrak{A}',$$

as required. $\square$

We are now ready to finish the proof of Theorem 3.2.1(i).

As a first step we combine the results of Lemmas 3.2.8, 3.2.10 and 3.2.11 to deduce that for each subset $\{\varphi_j\}_{1\le j\le a}$ of $\mathrm{Hom}_{\mathfrak{A}}(H^1(C),\mathfrak{A})$, each y_1 in $\mathrm{Ann}_{\mathfrak{A}'}(\mathrm{Ext}^2_{\mathfrak{A}'}(H^2(C'),\mathfrak{A}'))$ and each y_2 in $\mathrm{Ann}_{\mathfrak{A}}(\mathrm{Ext}^2_{\mathfrak{A}}(H^2(C),\mathfrak{A}))$ one has

$$(y_1y_2)^a\cdot(\wedge_{i=1}^{i=a}\varphi_i)(\eta_{\mathcal{X}}) = (y_1)^a\cdot(\wedge_{i=1}^{i=a}(y_2e_a\cdot\varphi_i))(\eta_{\mathcal{X}}) \subseteq \mathrm{Fit}^0_{\mathfrak{A}'}(H^2(C')/X).$$

Set $\mathfrak{A}^\dagger := \mathfrak{A}\cap\mathfrak{A}'$. Then to deduce the result of Theorem 3.2.1(i) from the above inclusion it suffices to prove that one has both

$$\mathfrak{A}^\dagger\cdot\mathrm{Fit}^0_{\mathfrak{A}'}(H^2(C')/X) \subseteq \mathrm{Fit}^a_{\mathfrak{A}}(H^2(C)) \ \text{ and } \ \mathfrak{A}^\dagger\cdot\mathrm{Fit}^0_{\mathfrak{A}'}(H^2(C')/X) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^2(C)'_{\mathrm{tor}}). \quad (3.19)$$

Since $\mathfrak{A}^\dagger\cdot\mathfrak{A}'\subseteq\mathfrak{A}$ the first of these inclusions follows directly from the fact that

$$\mathrm{Fit}^0_{\mathfrak{A}'}(H^2(C')/X) \subseteq \mathrm{Fit}^a_{\mathfrak{A}'}(H^2(C')) = \mathrm{Fit}^a_{\mathfrak{A}'}(\mathfrak{A}'\otimes_{\mathfrak{A}} H^2(C)) = \mathfrak{A}'\cdot\mathrm{Fit}^a_{\mathfrak{A}}(H^2(C)).$$

Here the inclusion is true because X is a free $\mathfrak{A}'$-module of rank a, the first equality because $H^2(C')$ is isomorphic to $\mathfrak{A}'\otimes_{\mathfrak{A}} H^2(C)$ and the second equality follows from a standard property of Fitting ideals under change of ring.

Next we note that the given surjective homomorphism of $\mathfrak{A}$-modules $H^2(C) \to H^2(C)'$ induces a surjective homomorphism of $\mathfrak{A}'$-modules $\varpi : H^2(C') \to \mathfrak{A}' \otimes_{\mathfrak{A}} H^2(C)'$ and we recall that, by assumption, the $\mathfrak{A}'$-module $\varpi(X)$ is free of rank a. This implies that

$$\mathrm{Fit}^0_{\mathfrak{A}'}(H^2(C')/X) \subseteq \mathrm{Ann}_{\mathfrak{A}'}((\mathfrak{A}' \otimes_{\mathfrak{A}} H^2(C)')/\varpi(X)) \subseteq \mathrm{Ann}_{\mathfrak{A}'}((\mathfrak{A}' \otimes_{\mathfrak{A}} H^2(C)')_{\mathrm{tor}}). \tag{3.20}$$

We now set $\mathfrak{A}^\sharp := \mathfrak{A} \cap \mathfrak{A}(1 - e_{(a)})$ and note that the tautological short exact sequence $0 \to \mathfrak{A}^\sharp \to \mathfrak{A} \to \mathfrak{A}' \to 0$ gives rise to an exact sequence of $\mathfrak{A}'$-modules of the form

$$(\mathfrak{A}^\sharp \otimes_{\mathfrak{A}} H^2(C)')_{\mathrm{tor}} \to H^2(C)'_{\mathrm{tor}} \to (\mathfrak{A}' \otimes_{\mathfrak{A}} H^2(C)')_{\mathrm{tor}}.$$

Since the first module in this sequence is clearly annihilated by $\mathfrak{A}^\dagger$ one therefore has

$$\mathfrak{A}^\dagger \cdot \mathrm{Ann}_{\mathfrak{A}'}((\mathfrak{A}' \otimes_{\mathfrak{A}} H^2(C)')_{\mathrm{tor}}) \subseteq \mathrm{Ann}_{\mathfrak{A}}(H^2(C)'_{\mathrm{tor}})$$

and this combines with (3.20) to imply the second inclusion of (3.19), as required to complete the proof of Theorem 3.2.1(i).

3.2.3 Proof of Theorem 3.2.1 (ii)

As preparation for the proof of Theorem 3.2.1(ii), in this section we show that strictly admissible complexes admit resolutions with certain useful features.

For any $\mathfrak{A}$-module M we set $M^* := \mathrm{Hom}_{\mathfrak{A}}(M, \mathfrak{A})$, endowed with the natural action of $\mathfrak{A}$. We recall that M is said to be 'reflexive' if the natural map $M \to (M^*)^*$ is bijective.

Lemma 3.2.12. For any object C of $D^{\mathrm{s}}(\mathfrak{A})$, the following claims are valid.

(i) The $\mathfrak{A}$-module $H^1(C)$ is reflexive.

(ii) Assume that the Euler characteristic of C in $K_0(\mathfrak{A})$ vanishes. Then for each separable subset $\mathcal{X}$ of $H^2(C)$, there exists an isomorphism in $D(\mathfrak{A})$ between C and a complex of the form $P \xrightarrow{d} P$, where P is a finitely generated free $\mathfrak{A}$-module, the first term is placed in degree one and both of the following conditions are satisfied.

(a) The natural map $P^* \to \ker(d) \cong H^1(C)$ is surjective.

(b) Set $a := |\mathcal{X}|$, denote the elements of $\mathcal{X}$ by $\{x_i\}_{1\le i\le a}$ and write d for the $\mathfrak{A}$-rank of P. Then $d \ge a$ and there exists an ordered basis $\{b_i\}_{1\le i\le d}$ of P such that $\mathrm{Im}(d)$ is contained in the $\mathfrak{A}$-submodule generated by $\{b_i\}_{a<i\le d}$ and for each i with $1 \le i \le a$ the natural map $P \to \mathrm{cok}(d) \cong H^2(C)$ sends b_i to x_i.

Proof. It is enough to prove claim (i) after localising at each prime ideal of R and so we may assume that C is represented by a complex of the form $P \to P$, where P is a finitely generated free $\mathfrak{A}$-module and the first term is placed in degree one.

For any complex C' in $D^{\mathrm{p}}(\mathfrak{A})$ we set $(C')^\dagger := R\,\mathrm{Hom}_{\mathfrak{A}}(C, \mathfrak{A}[-3])$.

Then, since P is both isomorphic to P^* and reflexive, the complex $C^\dagger$ belongs to $D^{\mathrm{s}}(\mathfrak{A})$ and the complex $(C^\dagger)^\dagger$ identifies with C.

In addition, by using the universal coefficient spectral sequence [61, Th. 10.90], one computes that $H^1(C) = H^1((C^\dagger)^\dagger)$ identifies with $H^2(C^\dagger)^*$ and that there is a natural exact sequence of $\mathfrak{A}$-modules

$$0 \to \mathrm{Ext}^1_{\mathfrak{A}}(H^2(C), \mathfrak{A}) \to H^2(C^\dagger) \to H^1(C)^* \to \mathrm{Ext}^2_{\mathfrak{A}}(H^2(C), \mathfrak{A}) \to 0.$$

In particular, since the modules $\mathrm{Ext}^1_{\mathfrak{A}}(H^2(C), \mathfrak{A})$ and $\mathrm{Ext}^2_{\mathfrak{A}}(H^2(C), \mathfrak{A})$ are both finite, the $\mathfrak{A}$-linear dual of this exact sequence induces an injective homomorphism from $(H^1(C)^*)^*$ to $H^2(C^\dagger)^* = H^1(C)$ that is inverse to the canonical map $H^1(C) \to (H^1(C)^*)^*$. This shows that $H^1(C)$ is reflexive, and hence proves claim (i).

To prove claim (ii) we first fix a surjective homomorphism of $\mathfrak{A}$-modules $\theta : P_1 \to H^1(C)^*$ where P_1 is finitely generated and free. Then, taking into account claim (i), the $\mathfrak{A}$-linear dual θ^* of θ gives an injective homomorphism from $H^1(C)$ to the (free) $\mathfrak{A}$-module $P_1^* \cong P_1$ with the property that its linear dual $(\theta^*)^*$ is the surjective homomorphism θ.

By a standard argument one now shows that C is isomorphic to a complex of the form $P_1 \xrightarrow{d_1} M$, where $\ker(d_1) = \mathrm{Im}(\theta^*)$ and the module M is finitely generated. Then, since P_1 is free and C belongs to $D^{\mathrm{p}}(\mathfrak{A})$ the module M has a finite projective resolution and hence, as $\mathfrak{A}$ has dimension one, there exists an exact sequence of finitely generated $\mathfrak{A}$-modules

$$0 \to P_2 \xrightarrow{d_2} P_3 \to M \to 0$$

where P_2 is projective and P_3 is free. It is then clear that C is isomorphic in $D(\mathfrak{A})$ to the complex $P_1 \oplus P_2 \to P_3$, where the first term is placed in degree one, the differential is $d := (d_1', d_2)$ with d_1' any lift of d_1 through the given surjection $P_2 \to M$ and the map θ^* induces an identification of $H^1(C)$ with $\ker(d) = \{(x, y) : x \in \mathrm{Im}(\theta^*), d_2(y) = -d_1'(x)\}$.

In particular, since the Euler characteristic of C in $K_0(\mathfrak{A})$ is assumed to vanish, the $\mathfrak{A}$-module $P_1 \oplus P_2$ must be isomorphic to the free module P_3.

Thus, if we set $P := P_1 \oplus P_2$, then at this stage we have shown C to be isomorphic to a complex $P \xrightarrow{d} P$ that has the property in claim (ii)(a).

Next we note that, since the $\mathfrak{A}$-module X generated by $\mathcal{X}$ is a direct summand of $H^2(C)$ we can fix a left inverse $H^2(C) \to X$ to the inclusion $X \subseteq H^2(C)$. Then, since X is free and the composite homomorphism $\pi : P \to \mathrm{cok}(d) \cong H^2(C) \to X$ is surjective, we obtain a direct sum decomposition of $\mathfrak{A}$-modules $P = \ker(\pi) \oplus X'$ in which $\ker(\pi)$ is free and X' is mapped bijectively by π to X.

Hence, since $\mathrm{Im}(d) \subseteq \ker(\pi)$, we obtain a basis of the sort required in claim (ii)(b) by taking $\{b_i\}_{a<i\leq d}$ to be any basis of $\ker(\pi)$ and, for each $1 \leq i \leq a$, defining b_i to be the unique

element of X' with $\pi(b_i) = x_i$. $\square$

Now we proceed to finish the proof of Theorem 3.2.1 (ii).

As a first step we note that, since $\mathcal{X}$ is now assumed to be separable it spans a free $\mathfrak{A}$-module X of rank a. In this case it is thus clear that $e_{C,(a)} = 1$ (so $\mathfrak{A}' = \mathfrak{A}$) and hence that Theorem 3.2.1(ii) with $x = 1$ gives an inclusion $y^a \cdot I(\eta) \subseteq \mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))$.

It therefore only remains to prove that $\mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))$ is contained in $I(\eta)$. By Lemma 3.2.12, one can choose a distinguished representative of C of the form $P \xrightarrow{\psi} P$ with the described properties. Consider the presentation of $H^2(C)$ given by $P \xrightarrow{\psi} P \xrightarrow{\pi} H^2(C)$. Define $\psi_i := \psi \circ b_i^*$ where b_i^* is the dual of b_i. Note that $\psi_i = 0$ for each $i \leq a$ because, by Lemma 3.2.12 (ii), $\mathrm{Im}(\psi)$ is contained in the $\mathfrak{A}$-module generated by $\{b_i\}_{a<i\leq d}$. Thus, $\mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))$ is generated by the minors $\{\det(\psi_i(b_{\sigma(k)})_{a<i,k\leq d}\}_{\sigma\in\mathfrak{S}_{d,a}}$.

In the sequel, we identify $\mathrm{Det}_{\mathfrak{A}}(C)$ with $\bigwedge^d_{\mathfrak{A}} P^* \otimes_{\mathfrak{A}} \bigwedge^d_{\mathfrak{A}} P$. Define $z_b = z\bigwedge_{i=1}^d b_i$ in $\bigwedge^d_{\mathfrak{A}} P$ with $z \in \mathfrak{A}^\times$ to be the pre-image of $\mathcal{L}^{-1}$ under the composite isomorphism of $\mathfrak{A}$-modules

$$\bigwedge\nolimits^d_{\mathfrak{A}} P \xrightarrow{\sim} \bigwedge\nolimits^d_{\mathfrak{A}} P \otimes \bigwedge\nolimits^d_{\mathfrak{A}} P^* = \mathrm{Det}^{-1}_{\mathfrak{A}}(C) \xrightarrow{\vartheta_\lambda} \mathfrak{A} \cdot \mathcal{L}^{-1},$$

where the first map sends each v to $v \otimes \bigwedge_{1\leq i\leq d} b_i^*$.

We consider the composition

$$\begin{aligned}\mathrm{Det}^{-1}_{\mathfrak{A}}(C) \subset \mathrm{Det}^{-1}_{A_E}(E \otimes_R C) &\xrightarrow{\times e_{C,a}} e_{C,a}\,\mathrm{Det}^{-1}_{A_E}(E \otimes_R C) \\ &\cong (e_{C,a}\,\mathrm{Det}_{A_E}(E \otimes_R H^1(C))) \otimes_{A_E} (e_{C,a}\,\mathrm{Det}^{-1}_{A_E}(E \otimes_R H^2(C)))\end{aligned}$$

where the inclusion is obvious and the final map is the 'passage to cohomology' isomorphism.

Then [16, Lem 4.3] in this setting implies that

$$(-1)^{a(d-a)} \left(\bigwedge\nolimits_{i=1}^{i=a} \lambda \circ \bigwedge\nolimits_{a<i\leq d} \psi_i(z_b)\right) \otimes \bigwedge\nolimits_{i=1}^{i=a} x_i^* = (e_{C,a} \cdot \mathcal{L}^{-1} \bigwedge\nolimits_{i=1}^{i=a} x_i) \otimes \bigwedge\nolimits_{i=1}^{i=a} x_i^*.$$

By the definition of the higher special element, we have $\eta = (-1)^{a(d-a)} \bigwedge_{a<i\leq d} \psi_i(z_b)$ and hence the explicit formula in [16, Prop. 4.1] implies that

$$\eta = (-1)^{a(d-a)} z \sum_{\sigma \in \mathfrak{S}_{d,a}} \mathrm{sgn}(\sigma) \det(\psi_i(b_{\sigma(k)})_{a<i,k\leq d}(b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(a)})$$

where $\mathfrak{S}_{d,a} = \{\sigma \in S_d\colon \sigma(1) < \cdots < \sigma(a) \text{ and } \sigma(a+1) < \cdots < \sigma(d)\}$. Therefore,

$$(b^*_{\sigma(1)} \wedge \cdots \wedge b^*_{\sigma(a)})(\eta) = \pm z \det(\psi_i(b_{\sigma(k)})_{a<i,k\leq d}. \tag{3.21}$$

Since P^* surjects onto $H^1(C)^*$ by construction, (3.21) implies that

$$z \det(\psi_i(b_{\sigma(k)}))_{a<i,k\leq d} \in I(\eta) = \{(\wedge_{i=1}^{i=a} \varphi_i)(\eta)\colon (\varphi_i)_i \in \mathrm{Hom}_{\mathfrak{A}}(H^1(C), \mathfrak{A})^a\}.$$

As z is unit in $\mathfrak{A}$, we have the inclusion that $\mathrm{Fit}^a_{\mathfrak{A}}(H^2(C)) \subset I(\eta)$. This finishes the proof of Theorem 3.2.1 (ii).

3.3 Structure of exterior biduals

In this section, we study the structure of the quotient of exterior power biduals of $H^1(C)$ by the submodule generated by the corresponding special elements under the assumption that $\mathfrak{A}$ is Gorenstein.

3.3.1 Statement of the main results

For an $\mathfrak{A}$-lattice X and an idempotent e of A we set

$$X^e := \{x \in X : e \cdot x = x \text{ in } F \otimes_R X\}$$

(so that $X^e = e \cdot X$ if e belongs to $\mathfrak{A}$). For any $\mathfrak{A}$-module X we write X_{tor} for the submodule comprising all R-torsion elements. For any element x of X we denote the $\mathfrak{A}$-submodule that x generates by $\langle x \rangle$.

The main algebraic result in this section is the following.

Theorem 3.3.1. Assume that $\mathfrak{A}$ is Gorenstein. Fix data $(C, \lambda, \mathcal{L}, \mathcal{X})$ as in Definition 3.1.3. Set $a := |\mathcal{X}|$, $e_a := e_{C,a}$, $e_{(a)} := e_{C,(a)}$ and $\mathfrak{A}' := \mathfrak{A} e_{(a)}$ and write η in place of $\eta_{(C,\lambda,\mathcal{L},\mathcal{X})}$.

Then for any elements x of $\mathfrak{A} \cap \mathfrak{A}'$ and y of $\mathrm{Ann}_{\mathfrak{A}}(\mathrm{Ext}^2_{\mathfrak{A}'}(\mathfrak{A}' \otimes_{\mathfrak{A}} H^2(C), \mathfrak{A}'))$ such that both xe_a and ye_a are invertible in Ae_a, there exists a canonical isomorphism of (finite) $\mathfrak{A}$-modules

$$\mathrm{Hom}_{\mathfrak{A}}\left(\left(\frac{\bigcap_{\mathfrak{A}}^a H^1(C)}{\langle xy^a \cdot \eta \rangle}\right)_{\mathrm{tor}}, \frac{A}{\mathfrak{A}}\right) \cong \left(\frac{\mathfrak{A}}{xy^a \cdot I(\eta)}\right)_{\mathrm{tor}},$$

and hence also a canonical short exact sequence of $\mathfrak{A}$-modules

$$0 \to \left(\frac{\mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))}{xy^a \cdot I(\eta)}\right)_{\mathrm{tor}} \to \mathrm{Hom}_{\mathfrak{A}}\left(\left(\frac{\bigcap_{\mathfrak{A}}^a H^1(C)}{\langle xy^a \cdot \eta \rangle}\right)_{\mathrm{tor}}, \frac{A}{\mathfrak{A}}\right) \to \left(\frac{\mathfrak{A}}{\mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))^{e_a}}\right)_{\mathrm{tor}} \to 0.$$

Remark 3.3.2. If $e_{(a)}$ belongs to $\mathfrak{A}$ (as is automatically the case, for example, if $\mathcal{X}$ is separable), then $\mathfrak{A} \cap \mathfrak{A}' = \mathfrak{A}'$ is Gorenstein and so $\mathrm{Ext}^1_{\mathfrak{A}'}(\mathfrak{A}' \otimes_{\mathfrak{A}} H^2(C), \mathfrak{A}')$ vanishes. In such a case we can therefore take $x = y = e_{(a)}$ and omit them from the statement of Theorem 3.3.1. In particular, if $\mathcal{X}$ is separable, then Theorem 3.2.1(ii) and Remark 3.2.3 combine to imply that $I(\eta) = \mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))$ (which equals to $\mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))^{e_a}$ in this case) and hence that there is an isomorphism of $\mathfrak{A}$-modules

$$\mathrm{Hom}_{\mathfrak{A}}\left((\bigcap\nolimits_{\mathfrak{A}}^a H^1(C)/\langle \eta \rangle)_{\mathrm{tor}}, A/\mathfrak{A}\right) \cong \left(\mathfrak{A}/\,\mathrm{Fit}^a_{\mathfrak{A}}(H^2(C))\right)_{\mathrm{tor}}.$$

3.3.2 Proof of Theorem 3.3.1

In this section, we prove Theorem 3.3.1

At the outset we note that, by its very definition, η has non-zero component at each simple component of Ae_a. As xe_a and ye_a are both assumed to be invertible in Ae_a, the product $xy^a \cdot \eta$ also has non-zero component at each simple component of Ae_a and consequently we have that $\langle xy^a \cdot \eta \rangle \cong \mathfrak{A}e_a$.

In the sequel, we denote $xy^a \cdot \eta$ by $\tilde{\eta}$. To prove the claimed short exact sequence, firstly we observe that there is a natural identification

$$\left(\left(\bigcap\nolimits_{\mathfrak{A}}^{a} H^1(C)\right)^{e_a}\right)^* \xrightarrow{\sim} e_a \cdot \left(\bigcap\nolimits_{\mathfrak{A}}^{a} H^1(C)\right)^* = e_a \cdot \bigwedge_{\mathfrak{A}}^{a} (H^1(C))^* \tag{3.22}$$

where the arrow follows from Proposition 2.5.2(iii). By definition

$$\bigwedge\nolimits_{\mathfrak{A}}^{a} (H^1(C))^* \cdot e_a(\tilde{\eta}) = \bigwedge\nolimits_{\mathfrak{A}}^{a} (H^1(C))^* \cdot (\tilde{\eta}) = I(\tilde{\eta}) = xy^a \cdot I(\eta),$$

the assignment $\Phi \in e_a \cdot \bigwedge_{\mathfrak{A}}^{a} (H^1(C))^*$ to $\Phi(\tilde{\eta})$ thus gives an isomorphism

$$\left(\left(\bigcap\nolimits_{\mathfrak{A}}^{a} H^1(C)\right)^{e_a}\right)^* \to I(\tilde{\eta}). \tag{3.23}$$

Similarly, we also have an isomorphism

$$\langle \tilde{\eta} \rangle^* \xrightarrow{\sim} \mathfrak{A}^{e_a} \tag{3.24}$$

given by sending $\Phi \in \langle \tilde{\eta} \rangle^*$ to $\Phi(\tilde{\eta})$.

Let $Q = (\bigcap_{\mathfrak{A}}^{a} H^1(C))^{e_a} / \langle \tilde{\eta} \rangle^*$. Since $\langle \tilde{\eta} \rangle \cong \mathfrak{A}e_a$, we have that Q is a finite module. By the finiteness of Q, we have that $\mathrm{Hom}_{\mathfrak{A}}(Q, \mathfrak{A}) = 0$ and $\mathrm{Ext}^1_{\mathfrak{A}}(Q, A) = Q^{\vee}$. On the other hand, since $\mathfrak{A}$ is Gorenstein, $\mathrm{Ext}^1_{\mathfrak{A}}(\langle \tilde{\eta} \rangle, \mathfrak{A}) = \mathrm{Ext}^1_{\mathfrak{A}}(\mathfrak{A}e_a, \mathfrak{A}) = 0$. Consequently, if we apply the functor $\mathrm{Hom}_{\mathfrak{A}}(-,\mathfrak{A})$ to the tautological exact sequence

$$0 \to \langle\tilde{\eta}\rangle \to (\bigcap\nolimits_{\mathfrak{A}}^{a} H^1(C))^{e_a} \to Q \to 0,$$

then we obtain another short exact sequence

$$0 \to ((\bigcap\nolimits_{\mathfrak{A}}^{a} H^1(C))^{e_a})^* \to \langle\tilde{\eta}\rangle^* \to Q^\vee \to 0.$$

Combining with the isomorphisms (3.23) and (3.24),

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ((\bigcap_{\mathfrak{A}}^{a} H^1(C))^{e_a})^* & \longrightarrow & \langle\tilde{\eta}\rangle^* & \longrightarrow & Q^\vee & \longrightarrow & 0 \\
 & & \cong\big\downarrow & & \cong\big\downarrow & & & & \\
0 & \longrightarrow & xy^a \cdot I(\eta) & \longrightarrow & \mathfrak{A}^{e_a} & \longrightarrow & \dfrac{\mathfrak{A}^{e_a}}{xy^a \cdot I(\eta)} & \longrightarrow & 0
\end{array}$$

By the definitions of the isomorphisms (3.23) and (3.24), the square in the above diagram is commutative and hence one has that

$$Q^\vee = \left(\frac{(\bigcap_{\mathfrak{A}}^{a} H^1(C))^{e_a}}{\langle\tilde{\eta}\rangle}\right)^\vee \cong \frac{\mathfrak{A}^{e_a}}{xy^a \cdot I(\eta)}. \tag{3.25}$$

By Theorem 3.2.1 (i), we have that $xy^a \cdot I(\eta) \subset \mathrm{Fit}_{\mathfrak{A}}^{a}(H^2(C))^{e_a}$, the proof is finished by combining (3.25) and the tautological exact sequence

$$0 \to \frac{\mathrm{Fit}_{\mathfrak{A}}^{a}(H^2(C))^{e_a}}{xy^a \cdot I(\eta)} \to \frac{\mathfrak{A}^{e_a}}{xy^a \cdot I(\eta)} \to \frac{\mathfrak{A}^{e_a}}{\mathrm{Fit}_{\mathfrak{A}}^{a}(H^2(C))^{e_a}} \to 0$$

together with an easy observation as stated in the following lemma.

Remark 3.3.3. If e_a belongs to $\mathfrak{A}$, then the isomorphism (3.25) implies $\left(\bigcap_{\mathfrak{A}}^{a} H^1(C)/\langle xy^a \cdot \eta\rangle\right)_{\mathrm{tor}}$ is a cyclic $\mathfrak{A}$-module. To deal with the general case we assume (after localising) that $\mathfrak{A}$ is a (one-dimensional) local ring with maximal ideal $\mathfrak{m}$ and write $g_{\mathfrak{A}}(N)$ for the minimal number of generators of an $\mathfrak{A}$-module N. In this case the supremum $\nu(\mathfrak{A})$ of $g_{\mathfrak{A}}(I)$ as I runs over all ideals of $\mathfrak{A}$ is known to be finite and explicitly bounded in terms of the Hilbert function of the graded module $\mathrm{gr}_{\mathfrak{m}}(\mathfrak{A})$ (cf. [66]). In this case, therefore, the isomorphism (3.25) implies $g_{\mathfrak{A}}\big(\big(\bigcap_{\mathfrak{A}}^{a} H^1(C)/\langle xy^a \cdot \eta\rangle\big)_{\mathrm{tor}}\big) \leq \nu(\mathfrak{A})$.

Chapter 4

Generalised Stark Elements for $\mathbb{G}_m$

In this chapter, we review the theory of generalised Stark elements for $\mathbb{G}_m$ developed by Burns, Kurihara and Sano. We shall see that, under the validity of a relevant case of the Equivariant Tamagawa Number Conjecture, these Stark elements can be viewed as examples of higher special elements of certain strictly admissible complexes.

At the outset, we fix an odd prime p. Let F/k be an abelian extension of number fields with Galois group G and let S be a finite set of places of k containing all the non-archimedean places and places that ramify in F and T be a finite set of places of k disjoint from S.

4.1 Rubin-Stark elements and their conjectures

In this section, we review the classical Rubin-Stark elements and their conjectures.

Recall from §2.4.2 that we write $\mathcal{O}_{F,S,T}^{\times}$ for the subgroup of $\mathcal{O}_{F,S}^{\times}$ comprising elements that are congruent to 1 modulo all places in T_F and $X_{F,S}$ is the kernel of the natural morphism $\bigoplus_{w\in S_F}\mathbb{Z}\to\mathbb{Z}$. In the setting of the Rubin-Stark conjecture, we impose the following two hypotheses.

(i) T is chosen such that $\mathcal{O}_{F,S,T}^{\times}$ is $\mathbb{Z}$-torsion free.

(ii) r is a non-negative integer such that there exists a proper subset $V \subset S$ of cardinality r with the property that every place $v \in V$ splits completely in F.

We will fix T and r that satisfy the above hypotheses for the rest of this section.

4.1.1 Equivariant L-functions

Here we recall the definition of the equivariant L-functions. For any character χ in $\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^{\times})$, we define the S-truncated T-modified L-function of a complex variable z by setting

$$L_{k,S,T}(\chi, z) := \prod_{v \in T}(1 - \chi(\mathrm{Fr}_v^{-1})Nv^{1-z}) \prod_{v \notin S}(1 - \chi(\mathrm{Fr}_v)Nv^{-z})^{-1}, \tag{4.1}$$

where Fr_v denotes the (arithmetic) Frobenius in G of any place of F above v and Nv is the cardinality of the residue field at v.

This function is meromorphic in z and for each pair of integers d and j, we write $L_{k,S,T}^{(d)}(\chi, j)$ for the coefficient of $(z - j)^d$ in the Laurent expansion of $L_{k,S,T}(\chi, z)$ about $z = j$. We then define $r_{\chi,S}(j)$ to be the smallest integer d for which $L_{k,S,T}^{(d)}(\chi, j) \neq 0$ and denote the corresponding coefficient by $L_{k,S,T}^{*}(\chi, j)$.

The $\mathbb{C}[G]$-valued (equivariant) L-function associated to the data $(F/k, S, T)$ is defined by setting

$$\theta_{F/k,S,T}(z) := \sum_{\chi \in \widehat{G}} L_{k,S,T}(\chi^{-1}, z)e_{\chi}$$

and has leading term at $z = j$ equal to

$$\theta_{F/k,S,T}^{*}(j) := \sum_{\chi \in \widehat{G}} L_{k,S,T}^{*}(\chi^{-1}, j)e_{\chi} \in \mathbb{C}[G]^{\times},$$

where $L_{k,S,T}^{*}(\chi^{-1}, j)$ denotes the leading term at $z = j$ of the function $L_{k,S,T}(\chi^{-1}, z)$.

Here we recall a result of Tate concerning the order of vanishing of the L-function at the value $z = 0$. For each χ belonging to $\widehat{G}$, we write $S_\chi := \{v \in S : G_v \subseteq \ker(\chi)\}$ where G_v is the decomposition group of the place v in G. We also write $\mathbf{1}$ for the trivial homomorphism $G \to \mathbb{Q}^{c\times}$.

Proposition 4.1.1. [76, Chap. I, Prop. 3.4] For each χ belonging to $\widehat{G}$,

$$r_{\chi,S}(0) = \dim_{\mathbb{C}}(e_\chi \mathbb{C}X_{K,S}) = \begin{cases} |S_\chi| & \text{if } \chi \neq \mathbf{1} \\ |S| - 1 & \text{if } \chi = \mathbf{1} \end{cases}. \tag{4.2}$$

By combining Proposition 4.1.1 with hypothesis (ii) imposed in the beginning of this section, we have that $r \leq r_{\chi,S}(0)$ for every $\chi \in \widehat{G}$ and hence $z^{-r}L_{k,S,T}(\chi, z)$ is holomorphic at $z = 0$. We then define the 'r-th Stickelberger element' by setting

$$\theta^{(r)}_{F/k,S,T}(0) := \lim_{z \to 0} \sum_{\chi \in \widehat{G}} z^{-r} L_{k,S,T}(\chi^{-1}, z) e_\chi.$$

Since this element is clearly invariant under the complex conjugation, we have that $\theta^{(r)}_{F/k,S,T}(0) \in \mathbb{R}[G]$.

4.1.2 The Rubin-Stark conjecture

The Dirichlet unit theorem implies that the homomorphism $\mathcal{O}^\times_{F,S} \to \mathbb{R} \cdot X_{F,S}$ sending u to $-\sum_{w \in S_F} \log|u|_w w$ induces, for each non-negative integer r, a canonical isomorphism of $\mathbb{R}[G]$-modules

$$\lambda_{F,S} : \mathbb{R} \otimes_{\mathbb{Z}} \bigwedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{F,S,T} \xrightarrow{\sim} \mathbb{R} \otimes_{\mathbb{Z}} \bigwedge^r_{\mathbb{Z}[G]} X_{F,S}. \tag{4.3}$$

By convention, we fix an order $V = \{v_1, v_2, ..., v_r\}$ where V is the set that occurs in hypothesis (ii) and take any place $v_0 \in S \backslash V$. For each v_i, we fix a place w_i of F above v_i. Now we are ready to define the Rubin-Stark elements.

Definition 4.1.2. The Rubin-Stark element with respect to the data $(F/k, S, T, V)$ is the unique element

$$\eta^V_{F/k,S,T} \in \mathbb{R} \otimes_{\mathbb{Z}} \bigwedge^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{F,S,T}$$

that satisfies

$$\lambda_{F,S}(\eta^V_{F/k,S,T}) = \theta^{(r)}_{F/k,S,T}(0) \cdot \bigwedge_{i=1}^{r}(w_i - w_0) \text{ in } \mathbb{R} \otimes_{\mathbb{Z}} \bigwedge^r_{\mathbb{Z}[G]} X_{F,S}.$$

Example 4.1.3.

(i) When $r = 0$ (and hence V is empty), $\lambda_{F,S}$ in (4.3) is just the identity map on $\mathbb{R}[G]$ and the Rubin-Stark element $\eta^V_{F/k,S,T}$ is the (T-modified) Stickelberger element $\theta_{F/k,S,T}(0)$ for F/k.

(ii) Let $k = \mathbb{Q}$ and $F = \mathbb{Q}(\mu_f)^+$ with conductor $f > 0$. Set $r = 1$, $V = \{\infty\}$ and fix w to be any place of F above ∞. If we write $\zeta_f = e^{2\pi\sqrt{-1}/f}$, then it was proven in [76, p. 79] that

$$L'_S(\chi, 0) = -\frac{1}{2}\sum_{\sigma\in G} \log|(1-\zeta_f^\sigma)(1-\zeta_f^{-\sigma})|_w \chi(\sigma). \tag{4.4}$$

We further set $S = \{p|f\} \cup \{\infty\}$ and suppose that T contains an odd prime. Recall that the T-modified cyclotomic unit is defined by setting $c_{F,T} := \delta_T \cdot 2^{-1}(1-\zeta_f)(1-\zeta_f^{-1}) \in \mathcal{O}^\times_{F,S,T}$ where $\delta_T := \prod_{\ell\in T}(1-\sigma_\ell^{-1}\cdot\ell)$ where σ_ℓ is the Frobenius element of ℓ. Then (4.4) is equivalent to the equality

$$\lambda_{F,S}(c_{F,T}) = \theta^{(1)}_{F/k,S,T}(0) \cdot (w - w_0). \tag{4.5}$$

Hence, this implies that the T-modified cyclotomic unit coincides with the Rubin-Stark element for the data $(\mathbb{Q}(\zeta_f)^+/\mathbb{Q}, \{p|f\} \cup \{\infty\}, T, \{\infty\})$.

(iii) If $k = \mathbb{Q}(\sqrt{-d})$ for some $d > 0$, then Stark [70] has shown that the Rubin-Stark element in this case is essentially the elliptic unit which is constructed by the torsion points of an elliptic curve with complex multiplication by $\mathcal{O}_k$. The key properties of elliptic units that are relevant to this context are nicely surveyed in the proof of [76, Chap. IV, Prop. 3.9].

The original Stark conjecture predicts the existence of some special elements in the exterior power of the ring of integers that would interpolate derivatives of Artin L-functions in the same way as the cyclotomic units interpolating that of Dirichlet L-function. In [63], Rubin suggested that it would be too optimistic to expect that such elements possess no denominators. He subsequently defined some larger lattices in terms of the exterior power bidual functor that we recalled in §2.6 and formulated explicit conjectures describing the integrality of 'Stark's elements' in terms of these lattices. The following is the central conjecture of [63, Conj B'].

Conjecture 4.1.4 (Rubin-Stark conjecture)**.** One has

$$\eta^V_{F/k,S,T} \in \bigcap_{\mathbb{Z}[G]}^{r} \mathcal{O}^\times_{F,S,T}.$$

Remark 4.1.5. For any fixed number field k, let $\mathcal{K}/k$ be (a possibly infinite) abelian p-extension such that all infinite places of k split completely in $\mathcal{K}$ and set $\Omega(\mathcal{K}/k)$ to be the set of intermediate fields in $\Omega(\mathcal{K}/k)$ which are finite over k. If the Rubin-Stark onjecture is valid for $(F/k, S, T, V)$ for any $F \in \Omega(\mathcal{K}/k)$, then the collection of elements $\{\eta^V_{F/k,S,T} : F \in \Omega(\mathcal{K}/k)\}$ forms an Euler system of rank r in the sense of Burns and Sano introduced in [23]. We remark that it was shown in loc. cit. for a higher rank Euler system to posses any meaningful arithmetic application, it was necessary for the elements in the system to be 'integral' in the sense of the Rubin-Stark conjecture.

4.1.3 Connection to the eTNC

In this section, we present the known cases of the Rubin-Stark conjecture by describing its relation with the Equivariant Tamagawa Number Conjecture (eTNC for short) formulated by Burns and Flach in [13]. To do this, we write $C_{F,S,T}$ for the complex $R\Gamma_T((\mathcal{O}_{F,S})_{\mathcal{W}}, \mathbb{G}_m)[-1]$ that occurs in the work of [16] whose main properties have been recalled in §2.4.1.

Since $\mathrm{Cl}_S^T(F)$ is finite, the exact sequence (2.10) and Proposition 2.4.2 imply that the Dirichlet regulator map induces an isomorphism of $\mathbb{R}[G]$-modules of the form

$$\lambda_{F,S} : \mathbb{R} \otimes_{\mathbb{Z}} H^1(C_{F,S,T}) \xrightarrow{\sim} \mathbb{R} \otimes_{\mathbb{Z}} H^2(C_{F,S,T}). \tag{4.6}$$

The eTNC for the pair $(h^0(\mathrm{Spec}(F), \mathbb{Z}[G])$ can be stated as follows.

Conjecture 4.1.6. $\theta^*_{F/k,S,T}(0)^{-1}$ is a characteristic element for the pair $(C_{F,S,T}, \lambda_{F,S})$.

In the sequel, we simply say that eTNC(F/k) holds if Conjecture 4.1.6 is valid. The following proposition recognises Rubin-Stark elements as the higher special elements that we introduced in §3 coming from the Weil-étale cohomology complex and Dirichlet regulator isomorphism.

Proposition 4.1.7. Assume eTNC(F/k) is valid. Then the Rubin-Stark element $\eta^V_{F/k,S,T}$ coincides with the higher special element associated with the data ($C_{F,S,T}$, $\lambda_{F,S}$, $\theta^*_{F/k,S,T}(0)^{-1}$, $\mathcal{X}_V$) where $\mathcal{X}_V = \{w_i - w_0 : 1 \leq i \leq r\}$. Moreover, the Rubin-Stark conjecture for the data $(F/k, S, T, V)$ is valid.

Proof. We abbreviate $C := C_{F,S,T}$ and set $r := |V|$. We also recall from §3.1 that $e_{C,r}$ is the sum of all primitive idempotents e_χ such that $\dim_{\mathbb{C}}(e_\chi \mathbb{C}H^2(C)) = r$. By the definition of $e_{C,r}$ and Proposition 4.1.1, we have that $e_{C,r} \cdot \theta^*_{F/k,S,T}(0) = \theta^{(r)}_{F/k,S,T}(0)$. Now the first assertion follows from simply comparing Definition 3.1.3 and Definition 4.1.2. The second assertion then follows from specialising Theorem 3.2.5 (i) in this setting. (Note that in this case x and y can be omitted because $e_{C,(r)} = 1$ and $\mathbb{Z}[G]$ is Gorenstein by [28, Cor. 10.29]. See Remark 3.2.2.) $\square$

Remark 4.1.8. The proof that the Rubin-Stark conjecture is a consequence of a relevant case of the Equivariant Tamagawa Number Conjecture was first offered by Burns in his seminal work [8] in which he has, indeed, derived connections between the later conjecture with many more existing conjectures/results regarding the arithmetic properties of the leading terms of the Artin L-functions. We also note that this proof has been simplified later by Burns, Kurihara and Sano in [16].

Here we record two important cases in which eTNC(F/k) is valid unconditionally.

Theorem 4.1.9. Conjecture 4.1.6 is valid in the following cases:

(i) F is abelian over $\mathbb{Q}$,

(ii) F/k is quadratic.

In particular, Proposition 4.1.7 is valid unconditionally in these cases.

Proof. The case (i) is the main result of Burns and Greither in [15] and of Flach [34]. The case (ii) is proven by Kim in [47]. $\square$

Remark 4.1.10. There are also partial results of Bley [4] for certain abelian extensions of imaginary quadratic fields, Buckingham [7] for certain biquadratic extensions and Johnston and Nickel [41, Cor. 4.8] for certain relative cyclic extensions.

4.2 Stark elements of arbitrary ranks and weights

4.2.1 T-modified cohomology

To define the generalised Stark elements, we shall first make some useful observations concerning earlier constructions of Burns, Kurihara and Sano. In the sequel, we assume that S also contains the p-adic places of k. Following the construction of [18], for any integer a, define $R\Gamma_T(\mathcal{O}_{F,S},\mathbb{Z}_p(a))$ to be the complex that lies in an exact triangle in $D(\mathbb{Z}_p[G])$ of the form

$$R\Gamma_T(\mathcal{O}_{F,S},\mathbb{Z}_p(a)) \to R\Gamma(\mathcal{O}_{F,S},\mathbb{Z}_p(a)) \to \bigoplus_{w\in T_F} R\Gamma(\kappa(w),\mathbb{Z}_p(a)) \tag{4.7}$$

and in each degree i the 'T-modified' étale cohomology of $\mathbb{Z}_p(a)$ is then defined by setting $H^i_T(\mathcal{O}_{F,S},\mathbb{Z}_p(a)) := H^i(R\Gamma_T(\mathcal{O}_{F,S},\mathbb{Z}_p(a)))$.

Now we define a complex $C_{F,S,T}(a)$ that lies in an exact triangle in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ of the form

$$C_{F,S,T}(a) \to R\operatorname{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{F,S},\mathbb{Z}_p(a)),\mathbb{Z}_p)[-3] \to \bigoplus_{w\in T_F} R\Gamma(\kappa(w),\mathbb{Z}_p(1-a)) \to \tag{4.8}$$

where $R\Gamma_c(\mathcal{O}_{F,S},\mathbb{Z}_p(a))$ is the complex of compactly supported étale cohomology over $\mathrm{Spec}(\mathcal{O}_{F,S})$ of the sheaf $\mathbb{Z}_p(a)$ defined in §2.4.2.

Remark 4.2.1. By Example 2.4.7 (iii) (also see [18, Lem 4.1]), $C_{F,S,T}(a)$ is acyclic outside degrees 1 and 2, that $H^1(C_{F,S,T}(a))$ identifies with $H^1_T(\mathcal{O}_{F,S},\mathbb{Z}_p(1-a))$ and that $H^2(C_{F,S,T}(a))$ lies in a split exact sequence of $\mathbb{Z}_p[G]$-modules

$$0 \to H^2_T(\mathcal{O}_{F,S},\mathbb{Z}_p(1-a)) \to H^2(C_{F,S,T}(a)) \to Y_F(-a) \to 0$$

where $Y_F(-a)$ is the $\mathbb{Z}_p[G]$-linear dual of the module $\bigoplus_{w\in S_\infty(F)} H^0(F_w,\mathbb{Z}_p(a))$, endowed with the natural action of G on $S_\infty(L)$.

Note that if T is empty, we suppress it from the notation as usual. The following observation will be used in later sections.

Lemma 4.2.2. The group $H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(a))$ is $\mathbb{Z}_p$-torsion-free in each of the following cases.

(i) $a = 0$.

(ii) $a = 1$ and $F^\times$ contains no element of order p.

(iii) T is non-empty.

Proof. Firstly, by Remark 4.2.1, the long exact sequence of the triangle (4.8) identifies $H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(a))$ with the kernel of the natural diagonal map

$$\Delta : H^1(\mathcal{O}_{F,S}, \mathbb{Z}_p(a)) \to \bigoplus_{w \in T_F} H^1(\kappa(w), \mathbb{Z}_p(a))$$

and so we may regard $H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(a))$ as a submodule of $H^1(\mathcal{O}_{F,S}, \mathbb{Z}_p(a))$.

For the case $a = 0$, since $H^1(\mathcal{O}_{F,S}, \mathbb{Z}_p)$ can be identified with $\mathrm{Hom}(\mathrm{Gal}(M_F^S/F), \mathbb{Z}_p)$, where M_F^S is the maximal algebraic extension of F unramified outside S, which is torsion-free, the submodule $H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p)$ is also torsion-free. This proves (i).

Now consider the case $a \neq 0$. In this case the space $H^0(\mathcal{O}_{F,S}, \mathbb{Q}_p(a))$ vanishes and so the natural exact sequence

$$0 \to H^0(\mathcal{O}_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(a)) \to H^1(\mathcal{O}_{F,S}, \mathbb{Z}_p(a)) \to H^1(\mathcal{O}_{F,S}, \mathbb{Q}_p(a))$$

identifies $H^1(\mathcal{O}_{F,S}, \mathbb{Z}_p(a))_{\mathrm{tor}}$ with $H^0(\mathcal{O}_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(a))$. In particular, if $a = 1$ and $F^\times$ contains no elements of order p, one has $H^0(\mathcal{O}_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)) = 0$. This proves (ii).

To prove (iii), suppose further that T is non-empty. Note that since S contains all the p-adic places, any place w in T is prime to p and hence $\mathbb{Z}_p(a)$ is an unramified module over F_w. Writing F_w^{un} for the maximal unramified extension of F_w in F_w^c, this fact implies that $H^1(F_w^{\mathrm{un}}, \mathbb{Z}_p(a))$ identifies with the (torsion-free) group $\mathrm{Hom}(\mathrm{Gal}(F_w^c/F_w^{\mathrm{un}}), \mathbb{Z}_p(a))$. The standard inflation-restriction exact sequence therefore identifies $H^1(\kappa(w), \mathbb{Z}_p(a))_{\mathrm{tor}}$ with $H^1(F_w, \mathbb{Z}_p(a))_{\mathrm{tor}}$ and hence also, by the same argument as above, with $H^0(F_w, \mathbb{Q}_p/\mathbb{Z}_p(a))$.

At this stage we know that $\ker(\Delta)_{\mathrm{tor}}$ identifies with the kernel of the diagonal map

$$H^0(\mathcal{O}_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(a)) \to \bigoplus_{w \in T_F} H^0(F_w, \mathbb{Q}_p/\mathbb{Z}_p(a))$$

and so it is enough to note that this map is obviously injective. $\square$

4.2.2 Period-regulator isomorphisms

In this section, we review the constructions of the period-regulator isomorphisms for (twisted) Tate motives, following [18]. The definitions of period-regulator isomorphisms are motivated directly by the constructions of Bloch and Kato in [3] in their seminal work on the Tamagawa Number Conjecture.

We write $S_{\mathbb{C}}(F)$ for the set of complex places of F and then for each integer j we set

$$S_\infty^j(F) := \begin{cases} S_\infty(F) \text{ if } j \text{ is even} \\ S_{\mathbb{C}}(F) \text{ if } j \text{ is odd} \end{cases}.$$

We define

$$Y_F(j) := \bigoplus_{w \in S_\infty(F)} H^0(F_w, \mathbb{Z}_p(j)) = \bigoplus_{w \in S_\infty^j(F)} \mathbb{Z}_p(j).$$

We fix a basis $\{w(j)\}_{w \in S_\infty^j(F)}$ over $\mathbb{Z}[G]$ of this module (as in [18, §2.1]) as follows. First we fix embeddings $\mathbb{Q}^c$ into the completions of F and set $\xi := (e^{2\pi\sqrt{-1}/p^n})_n \in \mathbb{Z}_p(1)$ and then define $w(j) := (w(j)_{w'})_{w'}$ where $w(j)_{w'} := \xi^{\otimes j}$ if $w' = w$ and 0 otherwise.

Remark 4.2.3. Note that there exists a comparison isomorphism of $\mathbb{Q}_p[G]$-modules

$$Y_F(j) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} H^0(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), H^0_B(\mathrm{Spec}F(\mathbb{C}), \mathbb{Q}(j))) \otimes_{\mathbb{Q}} \mathbb{Q}_p,$$

where $H^0_B(\mathrm{Spec}F(\mathbb{C}), \mathbb{Q}(j))$ is the 0-th Betti cohomology group.

For each place v in $S_\infty(k)$, we now fix a place w_v in $S_\infty(F)$ above v and then set $S_\infty(F)/G := \{w_v : v \in S_\infty(k)\}$.

We also now fix an idempotent ε of $\mathbb{Z}_p[G]$ such that $\varepsilon Y_L(-j)$ is a free $\mathbb{Z}_p[G]\varepsilon$-module admits a canonical basis of the form $\{\varepsilon \cdot w(-j) : w \in W_j^\epsilon\}$ with $W_j^\varepsilon := \{w \in S_\infty^j(L) \cap (S_\infty(L)/G) : \varepsilon \cdot w(-j) \neq 0\}$. Note that according to [18, Lem. 2.1], any primitive idempotent of $\mathbb{Z}_p[G]$ satisfies such a hypothesis.

We then set $r_j^\varepsilon := |W_j^\varepsilon|$ and write $\widehat{G}^\varepsilon$ for the subset of $\widehat{G}$ comprising characters χ with $\varepsilon \cdot e_\chi \neq 0$. We then define

$$\widehat{G}_j^\varepsilon := \{\chi \in \widehat{G}^\varepsilon : \dim_{\mathbb{C}_p}(e_\chi \mathbb{C}_p H^1(\mathcal{O}_{F,S}, \mathbb{Q}_p(1-j)) = r_j^\varepsilon\}$$

and obtain an idempotent of $\mathbb{Q}_p[G]$ by setting

$$\varepsilon_j = \sum_{\chi \in \widehat{G}_j^\varepsilon} e_\chi. \tag{4.9}$$

We can now recall that, for any given integer j, the 'j-th period-regulator isomorphism' is defined in [18, §2.2] to be the isomorphism of $\mathbb{C}_p[G]$-modules

$$\lambda_j : \varepsilon_j \mathbb{C}_p {\bigwedge}_{\mathbb{Z}_p[G]}^{r_j^\varepsilon} H^1(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j)) \xrightarrow{\sim} \varepsilon_j \mathbb{C}_p {\bigwedge}_{\mathbb{Z}_p[G]}^{r_j^\varepsilon} Y_F(-j),$$

that is explicitly specified as follows.

- If $j < 0$, then λ_j is the r_j^ε-th exterior power of the composite isomorphism of $\mathbb{C}_p[G]$-modules

$$\varepsilon_j\mathbb{C}_p H^1(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j)) \xrightarrow{\sim} \varepsilon_j\mathbb{C}_p K_{1-2j}(\mathcal{O}_F) \xrightarrow{\sim} \varepsilon_j\mathbb{C}_p H_B^0(\mathrm{Spec}(F), \mathbb{Q}(j))^+ \xrightarrow{\sim} \varepsilon_j\mathbb{C}_p Y_F(-j).$$

  Here the first isomorphism is the inverse of the Chern character isomorphism, the second map is (-1)-times the Borel regulator and the third map is the comparison isomorphism mentioned earlier.

- If $j = 0$, then λ_0 is the r_0^ε-th exterior power of the composite isomorphism of $\mathbb{C}_p[G]$-modules

$$\varepsilon_0\mathbb{C}_p H^1(\mathcal{O}_{F,S}, \mathbb{Z}_p(1)) \cong \varepsilon_0\mathbb{C}_p\mathcal{O}_{F,S}^\times \xrightarrow{\sim} \varepsilon_0\mathbb{C}_p Y_F(0).$$

  Here the first isomorphism is induced by Kummer theory and the second by Dirichlet's regulator isomorphism.

- If $j = 1$, then λ_1 is the composite isomorphism of $\mathbb{C}_p[G]$-modules

$$\varepsilon_1\mathbb{C}_p \,\mathrm{Det}_{\mathbb{C}_p[G]}(H^1(\mathcal{O}_{F,S}, \mathbb{Z}_p)) \xrightarrow{\sim} \varepsilon_1(\mathrm{Det}_{\mathbb{C}_p[G]}(\mathbb{C}_p \otimes F)^*) \otimes_{\mathbb{C}_p[G]} \mathrm{Det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p H_B^0(\mathrm{Spec}(F), \mathbb{Q})^+)) \xrightarrow{\sim} \varepsilon_1 \,\mathrm{Det}_{\mathbb{C}_p[G]}(\mathbb{C}_p Y_F(-1)).$$

  Here the first isomorphism is induced by global class field theory and the p-adic exponential map $F_w \to \mathbb{Q}_p\mathcal{O}_{F_w}^\times$ for $w \in S_p(F)$, the second isomorphism is induced by the comparison theorem mentioned earlier (see [18, §2.2.3] for the detailed construction).

- If $j > 1$, then λ_j is defined in essentially the same way as λ_1 except that the p-adic exponential map is now replaced by the more general Bloch-Kato exponential map for $\mathbb{Q}_p(j)$ over F_w.

Remark 4.2.4. In each case the definition of the idempotent ε_j ensures that the given maps produce the required bijections.

Remark 4.2.5. We consider the case when k is totally real and F is CM. Write $c \in G$ to be the complex conjugation and set $\varepsilon^+ = (1+c)/2$. If j is even, the rank of $\varepsilon^+ Y_F(-j)$ over $\varepsilon^+\mathbb{Z}_p[G]$ is equal to $r = [k : \mathbb{Q}]$.

(i) If $j < 0$, a theorem of Soulé implies that $\varepsilon_j^+ = \varepsilon^+$ (See [18, Rem. 2.4(i)]).

(ii) If $j = 0$, then [18, Rem. 2.4(i)] and Proposition 4.1.1 imply that $e_\psi \cdot \varepsilon_0^+ \neq 0$ if and only if $r_{\psi,S}(0) = r$. In other words, we have $\varepsilon_0^+ = \sum_{r_{\psi,S}(0)=r} e_\psi$.

(iii) If $j > 0$, we can take $\varepsilon_j^+ = \varepsilon^+$ in this case by the definition of λ_j and the fact that $\varepsilon^+ Y_F(1-j)$ vanishes (also see [18, Rem. 2.6]).

4.2.3 Generalised Stark elements

Now we are ready to define the generalised Stark elements.

Definition 4.2.6. Assume T is empty if $j = 1$. Then, in all cases, the Stark element of rank r_j^ε and weight $-2j$ for the data $(F/k, S, T, \varepsilon)$ is the unique element

$$\eta^\varepsilon_{F/k,S,T}(j) \in \varepsilon_j \mathbb{C}_p \bigwedge_{\mathbb{Z}_p[G]}^{r_j^\varepsilon} H^1(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j))$$

that satisfies

$$\lambda_j(\eta^\varepsilon_{F/k,S,T}(j)) = \varepsilon_j \cdot \theta^*_{F/k,S,T}(j) \cdot \bigwedge_{w \in W_j^\varepsilon} w(-j).$$

Remark 4.2.7.

(i) It is natural to regard $\eta^{\varepsilon}_{F/k,S,T}(j)$ as being of weight $-2j$ since $\theta^{*}_{F/k,S,T}(j)$ is the leading term at zero of the motive $h^0(\mathrm{Spec}(F))(j)$, regarded as defined over k and with coefficients $\mathbb{Q}[G]$ and this motive has weight $-2j$ in the sense of Deligne (see [40]).

(ii) If k is totally real and F is CM, then Remark 4.2.5 (ii) implies that the element $\eta^{\varepsilon^+}_{F/k,S,T}(0)$ coincides with the Rubin-Stark element in Definition 4.1.2 for the data $(F^+/k, S, T, S_\infty)$ after completion at p.

Example 4.2.8. Here we recall the definition of Deligne-Soulé cyclotomic elements (also see [18, §5], [31, §3]). Set $k = \mathbb{Q}$ and $F = \mathbb{Q}(\mu_f)$ of conductor $f > 0$. For simplicity we assume that $f \not\equiv 2 \pmod 4$. Having fixed an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, we then set $\zeta_m := \iota^{-1}(e^{2\pi\sqrt{-1}/m})$ for each positive integer m. Then for each positive integer m, define

$$c_{1-j}(\zeta_f)_m := \mathrm{Cor}_{\mathbb{Q}(\mu_{p^m f})/L}((1 - \zeta_{p^m f}) \otimes \zeta_{p^m}^{\otimes(-j)}) \in H^1(\mathcal{O}_{L,S}, \mathbb{Z}/p^m(1-j)).$$

Note that we have identified $(1 - \zeta_{p^m f}) \in \mathbb{Z}[\mu_{p^m f}, 1/p]^\times \cong H^1(\mathbb{Z}[\mu_{p^m f}, 1/p], \mathbb{Z}/p^m(1))$ by Kummer theory. The Deligne-Soulé cyclotomic element is defined to be

$$c_{1-j}(\zeta_f) := \varprojlim_m c_{1-j}(\zeta_f)_m \in H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)).$$

Let $S = \{\infty\} \cup \{l | f\}$ and $\varepsilon = e_j^+ := (1 + (-1)^j c)/2$ where $c \in G$ denotes the complex conjugation. Then in [18, §5.1] the authors have proven that $e_j^+ \cdot c_{1-j}(\zeta_f) = \eta^{\varepsilon}_{F/\mathbb{Q},S,\emptyset}(j)$. We remark that it was proven by using results of Beilinson and Hüber-Wildeshaus [43, Cor. 9.7] in the case $j < 0$ and by using Kato's explicit reciprocity [44, Th. 5.12] in the case $j > 0$.

4.3 The conjecture of Burns, Kurihara and Sano

4.3.1 Statement of the conjecture

In [18], the authors formulated a more refined conjecture which described the (conjectural) integrality of the Stark elements in terms of the initial Fitting ideal of a relevant cohomology module.

To formulate this conjecture, we define $C_{F,S,T}(j)$ to be any complex that fits in the exact triangle (4.8). In the rest of this chapter, we fix an idempotent $\varepsilon \in \mathbb{Z}_p[G]$ and an integer j. For simplicity, we abbreviate $r := r_j^{\varepsilon}$. Moreover, we set $C^{\varepsilon}_{F,S,T}(j) := \mathbb{Z}_p[G]\varepsilon \otimes_{\mathbb{Z}_p[G]} C_{F,S,T}(j)$. Recall from (3.4) that for any element $\eta \in \mathbb{C}_p \bigwedge^r_{\mathbb{Z}_p[G]} H^1(C^{\varepsilon}_{F,S,T}(j))$ one associates the ideal

$$I(\eta) := \{(\wedge_{i=1}^{i=r}\varphi_i)(\eta) : \varphi_i \in H^1(C^{\varepsilon}_{F,S,T}(j))^* \text{ for } 1 \leq i \leq r\}.$$

We write $I_{G,p}$ for the augmentation ideal of $\mathbb{Z}_p[G]$. Now we can recall the first central conjecture in [18].

Conjecture 4.3.1. [Burns, Kurihara, Sano] Assume $\varepsilon H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-torsion-free and, in addition, that ε belongs to $I_{G,p}$ if $j = 1$. Then one has

$$I(\eta^{\varepsilon}_{F/k,S,T}(j)) = \varepsilon \cdot \mathrm{Fit}^0_{\mathbb{Z}_p[G]}(H^2_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j))).$$

4.3.2 Connection to the eTNC

Here we recall the Equivariant Tamagawa Number Conjecture for the pair $(h^0(\mathrm{Spec}(F))(j), \mathbb{Z}_p[G]\varepsilon)$.

Conjecture 4.3.2. $\varepsilon \cdot \theta^*_{F/k,S,T}(j)^{-1}$ is a characteristic element for the pair $(C^{\varepsilon}_{F,S,T}(j), \lambda_j)$.

Proposition 4.3.3. Assume the validity of Conjecture 4.3.2. Then the Stark element $\eta^{\varepsilon}_{F/k,S,T}(j)$ coincides with the higher special element associated with the data $(C^{\varepsilon}_{F,S,T}(j)$, λ_j, $\varepsilon\theta^*_{F/k,S,T}(j)^{-1}$,

W_j^ε). Moreover, Conjecture 4.3.1 is valid.

Proof. We abbreviate $C := C^\varepsilon_{F,S,T}$ and $r = r^\varepsilon_j$. Since the Euler characteristic of the complex C vanishes, the definition of ε_j in (4.9) coincides with the definition of the idempotent $e_{C,r}$ in §3.1. Therefore, the first assertion follows from comparing Definition 3.1.3 and Definition 4.2.6 with respect to the prescribed data. The second assertion now follows directly from Theorem 3.2.1 (ii) (also see Remark 3.2.3). □

Remark 4.3.4. The claim that the Conjecture 4.3.1 follows as a consequence of the relevant case of the Equivariant Tamagawa Number Conjecture is first proven by Burns, Kurihara and Sano in [18]. In particular, after taking account of previous work of Burns and Greither [15] and of Flach [34], the authors were able in this way to deduce that Conjecture 4.3.1 is valid unconditionally if F is abelian over $\mathbb{Q}$

4.4 Generalised Kummer congruences

In 1851, Kummer discovered some congruences satisfied by the values of the Riemann zeta function at negative odd integers. One would like to generalise the congruences of Kummer to more general (abelian) L-functions and at other integer values. However, there are two major obstacles: firstly, it is easy for L-functions to vanish (for example, the Riemann zeta function vanishes at all negative even integers) and from the point of view of the Tamagawa Number Conjecture, it is more natural to discuss congruences of leading terms instead of just the values. Secondly, perhaps more seriously, the values of L-functions at positive integers are often believed to be transcendental so we cannot make sense of congruences between these values directly.

To overcome both issues, the insight of Burns, Kurihara and Sano is that despite the transcendacy of the leading terms of L-functions, the Stark elements, which can be regarded as their algebraic counterparts, are conjecturally integral in the sense of Conjecture 4.3.1. Therefore, it would be sensible to talk about congruences between the Stark elements (at different weights). This leads to another central conjecture of the authors in [18]. In this section, we review the statement of this congruence conjecture.

To do this we fix an integer j and a primitive idempotent ε of $\mathbb{Z}_p[G]$ and set $W = W_j^\varepsilon$ and $r = r_j^\varepsilon = |W|$. We also label, and hence order, the elements of W as $\{w_1, w_2, ..., w_r\}$.

We also fix a positive integer n with $\mu_{p^n} \subset F^\times$. We write $\chi_{cyc}^{F/k}$ for the associated cyclotomic character $G \to \mathrm{Aut}(\mu_{p^n}) \cong (\mathbb{Z}/p^n)^\times$ and for each integer a we define tw_a to be the ring automorphism of $\mathbb{Z}/p^n[G]$ that sends each element σ to $\chi_{cyc}^{F/k}(\sigma)^a\sigma$.

A key role is played by the $\mathrm{tw}_{j'-j}$-semilinear homomorphism

$$\mathrm{tw}_{j,j'}^{F/k} : {\bigcap}^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j)) \to {\bigcap}^r_{\mathbb{Z}/p^n[G]} H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}/p^n(1-j')) \tag{4.10}$$

that is defined as follows.

For each place w in $S_\infty(F)$ we fix an associated embedding $i_w : F \hookrightarrow \mathbb{C}$. For each integer u with $1 \le u \le r$ we then set

$$\xi_u := i_{w_u}^{-1}(e^{2\pi\sqrt{-1}/p^n}) \in \mu_{p^n}(F) = H^0(F, \mathbb{Z}/p^n(1))$$

and write c_u for the map

$$\mathrm{Hom}_{\mathbb{Z}/p^n[G]}(H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}/p^n(1-j')), \mathbb{Z}/p^n[G]) \to \mathrm{Hom}_{\mathbb{Z}/p^n[G]}(H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}/p^n(1-j)), \mathbb{Z}/p^n[G])$$

induced by taking cup product with $\xi_u^{\otimes(j-j')}$.

The map $\mathrm{tw}_{j,j'}^{F/k}$ is then defined to be the composite

$$\bigcap\nolimits^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j)) \to \bigcap\nolimits^r_{\mathbb{Z}/p^n[G]} H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}/p^n(1-j))$$
$$\to \bigcap\nolimits^r_{\mathbb{Z}/p^n[G]} H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}/p^n(1-j')).$$

Here the first map is induced by the natural map $H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j)) \to H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}/p^n(1-j))$ and the second is the $\mathbb{Z}/p^n[G]$-linear dual of the map

$$\bigwedge\nolimits^r_{\mathbb{Z}/p^n[G]} \mathrm{Hom}_{\mathbb{Z}/p^n[G]}(H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}/p^n(1-j')), \mathbb{Z}/p^n[G])$$
$$\to \bigwedge\nolimits^r_{\mathbb{Z}/p^n[G]} \mathrm{Hom}_{\mathbb{Z}/p^n[G]}(H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}/p^n(1-j)), \mathbb{Z}/p^n[G])$$

that sends each element $\wedge_{u=1}^{u=r} a_u$ to $\wedge_{u=1}^{u=r} c_u(a_u)$.

We can now state the Generalised Kummer Congruences conjecture formulated in [18].

Conjecture 4.4.1 (Generalised Kummer Congruences)**.** Fix an integer j'. Let $\overline{\varepsilon}$ be the image of ε under the natural projection $\mathbb{Z}_p[G] \to \mathbb{Z}/p^n[G]$ and write δ for the unique idempotent in $\mathbb{Z}_p[G]$ whose image in $\mathbb{Z}/p^n[G]$ coincides with $\mathrm{tw}_{j'-j}^{L/K}(\overline{\varepsilon})$.

Assume that the following conditions are satisfied:

(i) If $j = 1$ (respectively $j' = 1$), then ε (respectively δ) belongs to $I_{G,p}$ and T is empty;

(ii) The groups $\varepsilon \cdot H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j))$ and $\delta \cdot H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j'))$ are both torsion-free.

Then Conjecture 4.3.1 is true for both j and j' and there is an equality

$$\mathrm{tw}_{j,j'}^{L/k}(\eta^{\varepsilon}_{F/k,S,T}(j)) = \eta^{\delta}_{F/k,S,T}(j')$$

in the finite module $\bigcap^r_{\mathbb{Z}/p^n[G]} H^1(\mathcal{O}_{F,S}, \mathbb{Z}/p^n(1-j'))$.

Remark 4.4.2. It is shown in [18] that the equality in Conjecture 4.4.1 specialises to recover the classical congruences of Kummer concerning Bernouilli numbers and a certain 'Congruence Conjecture' of Solomon in [73] (which itself extends the explicit reciprocity law of Artin-Hasse and Iwasawa [38]).

In particular, by establishing such connections, the authors were able to obtain the following concrete evidence for Conjecture 4.4.1.

Theorem 4.4.3. [18, Thm 3.13] Assume that k is totally real and F is CM. Write c for the complex conjugation in $\mathrm{Gal}(F/k)$ and write $e_j^{\pm}$ for the idempotents $(1 \pm (-1)^j c)/2$ of $\mathbb{Z}_p[G]$.

(i) If $k = \mathbb{Q}$, then Conjecture 4.4.1 is true for all integers j and j' and with $\varepsilon = e_j^+$.

(ii) For all non-positive integers j and j', Conjecture 4.4.1 is valid with $\varepsilon = e_j^-$.

(iii) Conjecture 4.4.1 for the data $T = \emptyset$, $j = 0$, $\varepsilon = e_j^+$ and $j' = 1$ is a refinement of the Congruence Conjecture of Solomon [73].

In the sequel, we say that the Generalised Kummer Congruence is valid for the data $(F/k, \varepsilon, j, j')$ if Conjecture 4.4.1 is valid.

Chapter 5

Applications to Tate Motives

In this chapter, we specialise the theory of higher special elements developed in Chapter 3 to the p-adic representations that arise from (twisted) Tate motives. In this way, we both recover and refine the classical theory of abelian Stark conjectures (see §5.1) and the theory of generalised Stark elements developed by Burns, Kurihara and Sano in [18] (see §5.3.1). At the same time, we answer a question explicitly raised by both Washington and Lang regarding the Galois structure of global units modulo cyclotomic units in abelian fields (see §5.2), and also correct and significantly refine a result of El Boukhari [31] regarding the Galois structure of higher algebraic K-groups (see §5.3.2). Finally in §5.4 we formulate a new conjecture concerning the values of p-adic L-series at $s = 1$ and give evidence on this conjecture.

At the outset we fix a finite abelian extension F/k of number fields and set $G := \mathrm{Gal}(F/k)$. We also fix a finite set of places S of k that contains all places that are either archimedean or ramify in F/k and an auxiliary finite set of places T in k that is disjoint from S.

5.1 Higher Fitting ideals of Selmer modules and annihilators of ray class groups

In the next two sections, we discuss the special elements that are constructed from combining the 'Weil-étale cohomology complex' $C_{F,S,T} := R\Gamma_T((\mathcal{O}_{F,S})_{\mathcal{W}}, \mathbb{G}_m)[-1]$ from §2.4.1 and the Dirichlet regulator isomorphism in (4.6). Assume T is chosen such that $\mathcal{O}_{F,S,T}^{\times}$ is torsion-free and hence the complex $C_{F,S,T}$ is strictly admissible. Since the case $F = k$ is not of much interest we shall assume, only in this section, that G is not trivial.

We write $\mathbf{1}$ for the trivial homomorphism $G \to \mathbb{Q}^{c\times}$. For each non-negative integer a we write $\widehat{G}'_{S,a}$ for the subset of $\widehat{G} \setminus \{\mathbf{1}\}$ comprising homomorphisms ψ for which the set $S_\psi := \{v \in S : G_v \subseteq \ker(\psi)\}$ has cardinality a, where we write G_v for the decomposition subgroup in G of a place v of k. We then set

$$\widehat{G}_{S,a} := \begin{cases} \widehat{G}'_{S,a} \cup \{\mathbf{1}\}, & \text{if } a = |S| - 1, \\ \widehat{G}'_{S,a}, & \text{if } a \neq |S| - 1, \end{cases}$$

write $\widehat{G}_{S,(a)}$ for the union $\bigcup_{a' \geq a} \widehat{G}_{S,a'}$ and define idempotents of $\mathbb{Q}[G]$ by setting

$$e_{S,a} := \sum_{\psi \in \widehat{G}_{S,a}} e_\psi \quad \text{and} \quad e_{S,(a)} := \sum_{\psi \in \widehat{G}_{S,(a)}} e_\psi = \sum_{a' \geq a} e_{S,a'}.$$

In the next result, we denote the involution of $\mathbb{Z}[G]$ that inverts elements of G by $x \mapsto x^{\#}$. We will also make use of the Selmer module $\mathrm{Sel}_S^T(F)$ whose definition is recalled in Remark 2.4.1.

Theorem 5.1.1. Let $\mathcal{L}$ be a characteristic element for the pair $(C_{F,S,T}, \lambda_{F,S})$. Then for any non-negative integer a, any subset S_a of S that has cardinality at least $a + 1$ and contains

$\bigcup_{\psi\in\widehat{G}'_{S,a}} S_\psi$, any x in $\mathbb{Z}[G]\cap\mathbb{Z}[G]e_{S,(a)}$ and any Φ in $\bigwedge^a_{\mathbb{Z}[G]}\mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T},\mathbb{Z}[G])$ one has

$$\Phi(x\cdot\lambda^{-1}_{F,S}(e_{S,a}\cdot\mathcal{L}^{-1}\cdot\bigwedge\nolimits^a_{\mathbb{Z}[G]}X_{F,S}))\subseteq\mathrm{Fit}^a_{\mathbb{Z}[G]}(\mathrm{Sel}^T_S(F))^{\#}\cap\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S_a}(F)).$$

Proof. We abbreviate $C = C_{F,S,T}$, $\mathfrak{A} := \mathbb{Z}[G]$ and $\mathfrak{A}' := \mathbb{Z}[G]e_{S,(a)}$. First we prove that $e_{S,a} = e_{C,a}$ (see §3 for the definition of $e_{C,a}$). To show this, we first recall Proposition 4.1.1 that for each χ belonging to $\widehat{G}$,

$$\dim_{\mathbb{C}}(e_\chi\mathbb{C}X_{F,S}) = \begin{cases}|S_\chi| & \text{if } \chi\neq\mathbf{1}\\ |S|-1 & \text{if } \chi=\mathbf{1}\end{cases}. \tag{5.1}$$

Therefore, $e_\chi\cdot e_{S,a}\neq 0$ if and only if $\dim_{\mathbb{C}}(e_\chi\mathbb{C}X_{F,S}) = a$. Since $H^2(C) = \mathrm{Sel}^T_S(F)^{\mathrm{tr}}$ and the exact sequence (2.10) implies that $\mathbb{C}X_{F,S} = \mathbb{C}\mathrm{Sel}^T_S(F)^{\mathrm{tr}}$, we have $e_{S,a} = e_{C,a}$. In the rest, we abbreviate this idempotent as e_a.

Now we let $\mathcal{X}$ to be any subset of $H^2(C)_{\mathrm{tf}} = X_{F,S}$ (and set $a := |\mathcal{X}|$). We are going to specialise Theorem 3.2.1 (i) to the data $(C, \lambda_{F,S}, \mathcal{L}, \mathcal{X})$. Since $\mathbb{Z}[G]$ is Gorenstein, we can take $y = 1$ in the statement of Theorem 3.2.1(i) (see Remark 3.2.2). Write $\eta_{\mathcal{X}} = e_{S,a}\cdot\mathcal{L}^{-1}\cdot\lambda^{-1}_{F,S}(\bigwedge_{v\in\mathcal{X}} v)$. Theorem 3.2.1 (i) directly implies that for any $\Phi\in\bigwedge^a_{\mathbb{Z}[G]}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}^\times_{F,S,T},\mathbb{Z}[G])$, $\Phi(x\cdot\eta_{\mathcal{X}})$ belongs to $\mathrm{Fit}^a_{\mathfrak{A}}(\mathrm{Sel}^T_S(F)^{\mathrm{tr}})$. Then we note that in [16, Lem 2.8] it was proven that $\mathrm{Fit}^a_{\mathfrak{A}}(\mathrm{Sel}^T_S(F)^{\mathrm{tr}}) = \mathrm{Fit}^a_{\mathfrak{A}}(\mathrm{Sel}^T_S(F))^{\#}$. Therefore, we have $\Phi(x\cdot\eta_{\mathcal{X}})\in\mathrm{Fit}^a_{\mathfrak{A}}(\mathrm{Sel}^T_S(F))^{\#}$.

Now it remains to prove that

$$\Phi(x\cdot\lambda^{-1}_{F,S}(e_a\cdot\mathcal{L}^{-1}\cdot\bigwedge\nolimits^a_{\mathbb{Z}[G]}X_{F,S}))\subseteq\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S_a}(F)).$$

To do so, we will apply Theorem 3.2.1 (i) to the dual complex $C^* := \mathrm{RHom}_{\mathbb{Z}}(C,\mathbb{Z})[-2]$. The dual complex C^* is strictly admissible such that $H^1(C^*) = \mathrm{Hom}_{\mathbb{Z}}(X_{F,S},\mathbb{Z})$ and $H^2(C^*) = \mathrm{Sel}^T_S(F)$. Then $\lambda_{F,S}$ induces an isomorphism of $\mathbb{C}[G]$-modules $\lambda^*_{F,S} : \mathbb{C}H^1(C^*)\to\mathbb{C}H^2(C^*)$.

By an argument of [16, Prop. 3.4], if $\mathcal{L}$ is a characteristic element of C, then $\mathcal{L}^{\#}$ is a characteristic element of its dual complex C^*. Now Theorem 3.2.1 (i) specialises to imply that for any $x \in \mathbb{Z}[G] \cap \mathbb{Z}[G]e_{S,(a)}$, $\{c_i\}_{i=1}^a \subseteq \mathrm{Hom}_{\mathbb{Z}[G]}(H^1(C^*), \mathbb{Z}[G])$ and $\{\phi_i\}_{i=1}^a \subseteq \mathrm{Sel}_S^T(F)_{\mathrm{tf}}$,

$$x \cdot e_{S,a}\mathcal{L}^{-1,\#} \wedge_{i=1}^{i=a} c_i(\wedge_{i=1}^{i=a} \lambda_{F,S}^{*,-1}(\phi_i)) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_{S_a}^T(F)) \tag{5.2}$$

and hence $x \cdot e_{S,a}\mathcal{L}^{-1,\#} \det((\phi_i \circ \lambda_{F,S}^{-1})(c_j))_{i,j} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_{S_a}^T(F)^\vee)$. For any finitely-generated projective $\mathbb{Z}[G]$-modules P, there is a natural identification $\mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \cong \mathrm{Hom}_G(P, \mathbb{Z}[G])^{\#}$ where P acts on $\mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ contragradiently and on $\mathrm{Hom}_G(P, \mathbb{Z}[G])$ by right multiplication. Using this identification, the containment (6.10) implies that for any $\{b_i\}_{i=1}^a \subset X_{F,S}$, any $\{\psi_i\}_{i=1}^a \subseteq \mathrm{Hom}_G(\mathcal{O}_{F,S,T}^\times, \mathbb{Z}[G])$, one has

$$x^{\#} \cdot e_{S,a}\mathcal{L}^{-1,\#}(\det((\psi_i \circ \lambda_{F,S}^{-1})(b_j))_{i,j})^{\#} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_{S_a}^T(F)^\vee) \tag{5.3}$$

(note that we can freely replace x by $x^{\#}$ because the containment (6.10) is valid for any $x \in x \in \mathbb{Z}[G] \cap \mathbb{Z}[G]e_{S,(a)}$). Now note that the element appears in (5.3) equals to $(x \cdot e_{S,a}\mathcal{L}^{-1,\#} \wedge_{i=1}^{i=a} \psi_i \wedge_{i=1}^{i=a} \lambda_{F,S}^{-1}(b_i))^{\#}$ and hence we have

$$\Phi(x \cdot \lambda_{F,S}^{-1}(e_{S,a} \cdot \mathcal{L}^{-1} \cdot \bigwedge\nolimits_{\mathbb{Z}[G]}^a X_{F,S})) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_{S_a}^T(F)^\vee)^{\#}.$$

Since $\mathrm{Cl}_{S_a}^T(F)^\vee$ is a finite $\mathbb{Z}[G]$-module, we have $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_{S_a}^T(F)^\vee)^{\#} = \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_{S_a}^T(F))$. This finishes the proof.

□

Corollary 5.1.2. Assume that eTNC(F/k) is valid. Then for any non-negative integer a, any subset S' of S that has cardinality at least $a+1$ and contains $\bigcup_{\psi \in \widehat{G}'_{S,a}} S_\psi$, any x in

$\mathbb{Z}[G] \cap \mathbb{Z}[G]e_{S,(a)}$, any $\xi \in \bigwedge^a_{\mathbb{Z}[G]} X_{F,S}$ and any Φ in $\bigwedge^a_{\mathbb{Z}[G]} \mathrm{Hom}_G(\mathcal{O}^\times_{F,S,T}, \mathbb{Z}[G])$ one has

$$x \cdot \theta^{(a)}_{F/k,S,T}(0) \cdot \Phi(\lambda^{-1}_{F,S}(\xi)) \in \mathrm{Fit}^a_{\mathbb{Z}[G]}(\mathrm{Sel}^T_S(F))^{\#} \cap \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{S'}(F)).$$

In particular, the above containment is unconditional if F is abelian over $\mathbb{Q}$ or $[F : k] = 2$.

Proof. The validity of eTNC(F/k) (Conjecture 4.1.6) says that $\theta^*_{F/k,S,T}(0)^{-1}$ is a characteristic element for the pair $(C_{F,S,T}, \lambda_{F,S})$. By the proof of Proposition 4.1.7, one has that $e_{C,a}\theta^*_{F/k,S,T}(0)^{-1} = \theta^{(a)}_{F/k,S,T}(0)^{-1}$. Hence, the claimed result follows directly from Theorem 5.1.1. The last assertion follows from the known validity of eTNC(F/k) (see Theorem 4.1.9). □

Remark 5.1.3. In [19] Burns and Livingstone Boomla have recently proposed a refinement of higher-order abelian Stark conjectures that simultaneously improves all of the existing conjectures of this sort that are due, amongst others, to Stark [70], to Rubin [63], to Popescu [59], to Emmons and Popescu [32] and to Vallieres [77]. We remark that the containment described in Corollary 5.1.2 is actually more general than that predicted by Burns and Livingstone Boomla in [19, Conj. 4.3] because in loc. cit. the authors only consider elements of $\bigwedge^a_{\mathbb{Z}[G]} X_{F,S}$ of the form $\bigwedge_{v\in I}(w_v - w_I)$ where I is any subsets of $\bigcup_{\psi\in\widehat{G}'_{S,a}} S_\psi$ of cardinality a, w_v is any place of F above $v \in I$ and w_I is any choice of place of F above a place in $S \setminus I$.

5.2 On questions of Lang and Washington

To study the Galois structure of the quotient of the exterior bidual of the group of global units by the group of higher special elements, we specialise Theorem 3.3.1 to the setting of §5.1 in the case $F = \mathbb{Q}(\zeta_f)^+$ for some integer $f > 2$ and $k = \mathbb{Q}$. In [49, p. 260] Lang explicitly comments that the module structure of the quotient of the group of units by the group of cyclotomic units is a 'mystery'. This issue is partly addressed by the main result in this section.

To do this we set $S = \{\infty\} \cup \{\ell | f\}$ and let T be such that $\mathcal{O}_{F,S,T}^\times$ is $\mathbb{Z}$-torsion free. Recall from Example 4.1.3 (ii) that the T-modified cyclotomic unit $c_{F,T} := \delta_T \cdot 2^{-1}(1-\zeta_f)(1-\zeta_f^{-1}) \in \mathcal{O}_{F,S,T}^\times$ where $\delta_T := \prod_{\ell \in T}(1 - \sigma_\ell \cdot \ell)$ where σ_ℓ is the Frobenius element of ℓ.

Theorem 5.2.1. Set $F := \mathbb{Q}(\zeta_f)^+$ and $G := \mathrm{Gal}(F/\mathbb{Q})$. Then there exists a canonical isomorphism of G-modules

$$\mathrm{Hom}_{\mathbb{Z}}\left((\mathcal{O}_{F,S,T}^\times/\langle c_{F,T}\rangle)_{\mathrm{tor}}, \mathbb{Q}/\mathbb{Z}\right) \cong (\mathbb{Z}[G]/\operatorname{Fit}_G^1(\mathrm{Sel}_S^T(F)^{\mathrm{tr}}))_{\mathrm{tor}}$$

where the Pontryagin dual is endowed with the natural (rather than contragredient) action of G.

Proof. We abbreviate $C := C_{F,S,T}$. Let $V = \{\infty\} \subsetneq S$ and define $\mathcal{X}_V = \{w_1 - w_0\}$ as in Proposition 4.1.7. By combining Proposition 4.1.7 and Theorem 4.1.9, we know that the higher special element associated with the data $(C, \lambda_{F,S}, \theta_{F,S,T}^*(0), \mathcal{X}_V)$ coincides with the Rubin-Stark element for the data $(F/\mathbb{Q}, S, T, \{\infty\})$ unconditionally in this case. Now the proof finishes by specialising the above data to Theorem 3.3.1, combined with the fact that in this case the Rubin-Stark element is just the T-modified cyclotomic unit (See Example 4.1.3 (ii)). $\square$

Remark 5.2.2.

(i) Theorem 5.2.1 gives a natural isomorphism from the Pontryagin dual on the left, regarded as endowed with the contragredient action of G, to the torsion subgroup of the quotient $\mathbb{Z}[G]/\operatorname{Fit}_G^1(\mathrm{Sel}_S^T(F)^{\mathrm{tr}})$, regarded as endowed with the action of G obtained by composing the natural action with the involution $x \mapsto x^\#$ of $\mathbb{Z}[G]$ that inverts elements of G. The latter module is isomorphic to the torsion subgroup of $\mathbb{Z}[G]/\operatorname{Fit}_G^1(\mathrm{Sel}_S^T(F)^{\mathrm{tr}})^\#$, regarded as endowed with the natural action of G, and from [16, Lem. 2.8] one knows that $\mathrm{Fit}^1_G(\mathrm{Sel}^T_S(F)^{\mathrm{tr}})^{\#} = \mathrm{Fit}^1_G(\mathrm{Sel}^T_S(F))$, where the Selmer group $\mathrm{Sel}^T_S(F)$ is as defined in Remark 2.4.1. Taken together, these facts combine to give a perfect G-invariant pairing of the form

$$(\mathcal{O}^\times_{F,S,T}/\langle c_{F,T}\rangle)_{\mathrm{tor}} \times (\mathbb{Z}[G]/\mathrm{Fit}^1_G(\mathrm{Sel}^T_S(F)))_{\mathrm{tor}} \to \mathbb{Q}/\mathbb{Z}. \tag{5.4}$$

To be explicit, if u and v belong to $(\mathcal{O}^\times_{F,S,T}/\langle c_{F,T}\rangle)_{\mathrm{tor}}$ and $(\mathbb{Z}[G]/\operatorname{Fit}^1_G(\mathrm{Sel}^T_S(F)))_{\mathrm{tor}}$ respectively and one chooses representative elements in $(\mathcal{O}^\times_{F,S,T})^{e_1}$ and $\mathbb{Z}[G]^{e_1}$ for u and v, then the pairing in (5.4) sends (u,v) to $v\cdot c^*_{F,T}(u)$, where $c^*_{F,T}$ is the element of $\langle c_{F,T}\rangle^*$ that is dual to $c_{F,T}$.

(ii) Assume now that $f = p^n$ for some prime p. In this case the quotient $\mathcal{O}^\times_{F,S,T}/\langle c_{F,T}\rangle$ is finite and the G-module $X_{F,S}$ is free of rank one so that $e_1 = 1$ and the exact sequence (2.10) implies that $\mathrm{Fit}^0_G(\mathrm{Cl}^T_S(F)) = \mathrm{Fit}^1_G(\mathrm{Sel}^T_S(F)^{\mathrm{tr}})$. Theorem 5.2.1 therefore gives an isomorphism of (cyclic) G-modules $(\mathcal{O}^\times_{F,S,T}/\langle c_{F,T}\rangle)^\vee \cong \mathbb{Z}[G]/\operatorname{Fit}^0_G(\mathrm{Cl}^T_S(F))$.

This isomorphism resolves the problem explicitly raised by Washington in [78, Rem. following Th. 8.2] of establishing a precise connection in this case between the Galois structures of $\mathcal{O}^\times_{F,S,T}/\langle c_{F,T}\rangle$ and $\mathrm{Cl}^T_S(F)$.

5.3 Other weights

In this section, we study the higher special elements associated with the p-adic representation $\mathbb{Z}_p(j)$ with $j \notin \{0,1\}$. By specialising our theory in these cases, we will simultaneously recover the recent theory of Burns, Kurihara and Sano on generalised Stark elements in [18] and improve a result of El Boukhari in [31] regarding the Galois structure of the higher K-groups.

At the outset we fix p to be an odd prime and $\varepsilon \in \mathbb{Z}_p[G]$ to be any idempotent. In this section, we assume S also contains all the p-adic places of k. We set $C_{F,S,T}(j)$ to be the T-modified cohomology complex defined in §4.2.1 and define

$$C^{\varepsilon}_{F,S,T}(j) := \mathbb{Z}_p[G]\varepsilon \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C_{F,S,T}(j).$$

In addition, we fix the set T such that $\varepsilon H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free (we remark that this hypothesis is very mild according to Lemma 4.2.2).

5.3.1 On the generalised Stark elements

In the next result, we let λ be an isomorphism $EH^1(C^{\varepsilon}_{F,S,T}(j)) = EH^2(C^{\varepsilon}_{F,S,T}(j))$ for some algebraic extension E over $\mathbb{Q}_p$ (which exists by [18, Lem. 4.1]).

Theorem 5.3.1. Assume $j \notin \{0, 1\}$, fix a characteristic element $\mathcal{L}$ for $(C = C^{\varepsilon}_{F,S,T}(j), \lambda)$ and write r for the rank of the $\mathbb{Z}_p[G]\varepsilon$-module $\varepsilon \cdot Y_F(-j)$.

Then for any separable subset $\mathcal{X}$ of $H^2(C)_{\mathrm{tf}}$ with $|\mathcal{X}| = a$, there exists a canonical isomorphism of $\mathbb{Z}_p[G]\varepsilon$-modules

$$\left(\frac{\bigcap^a_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j))}{\langle \eta_{\mathcal{X}} \rangle}\right)^{\vee}_{\mathrm{tor}} \cong \left(\frac{\mathbb{Z}_p[G]}{\mathrm{Fit}^{a-r}_{\mathbb{Z}_p[G]}(H^2_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j)))^{\varepsilon_a}}\right)_{\mathrm{tor}}.$$

where $\eta_{\mathcal{X}}$ is the higher special element associated with the data $(C, \lambda, \mathcal{L}, \mathcal{X})$ and $\varepsilon_a := \varepsilon \cdot e_{C,a}$.

Proof. We abbreviate $C^{\varepsilon}_{F,S,T}(j)$ to C, $e_{C,a}$ to e_a and $H^i_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1-j))$ to H^i_T for $i = 1, 2$. To prove the claimed isomorphism, we specialise Theorem 3.3.1 to the data $(C, \lambda, \mathcal{L}, \mathcal{X})$. This yields

$$\left(\frac{\bigcap^a_{\mathbb{Z}_p[G]} H^1(C)}{\langle \eta_{\mathcal{X}} \rangle}\right)^{\vee}_{\mathrm{tor}} \cong \left(\frac{\mathbb{Z}_p[G]\varepsilon}{\mathrm{Fit}^{a}_{\mathbb{Z}_p[G]}(H^2(C))^{e_a}}\right)_{\mathrm{tor}}. \tag{5.5}$$

By Remark 4.2.1, we have $H^1(C) = \varepsilon H^1_T$ and $H^2(C) = \varepsilon H^2_T \oplus \varepsilon Y_F(-j)$. Since $\varepsilon \cdot \eta_{\mathcal{X}} = \eta_{\mathcal{X}}$ and $\varepsilon H^2(C) = H^2(C)$, Lemma 2.5.3 implies that $(\bigcap^a_{\mathbb{Z}_p[G]} \varepsilon H^1_T)/\langle \eta_{\mathcal{X}} \rangle$ (resp.

$\mathbb{Z}_p[G]\varepsilon/\operatorname{Fit}^a_{\mathbb{Z}_p[G]}(H^2(C))^{e_a})$ equals to the torsion subgroup of $(\bigcap^a_{\mathbb{Z}_p[G]} H^1_T)/\langle\eta_{\mathcal{X}}\rangle$ (resp. $\mathbb{Z}_p[G]/\operatorname{Fit}^a_{\mathbb{Z}_p[G]}(H^2(C))^{e_a})$. Moreover, since the $\mathbb{Z}_p[G]\varepsilon$-rank of the module $\varepsilon\cdot Y_F(-j)$ equals to r, by using the standard property of Fitting ideals, one has

$$\operatorname{Fit}^a_{\mathbb{Z}_p[G]\varepsilon}(H^2(C)) = \operatorname{Fit}^{a-r}_{\mathbb{Z}_p[G]\varepsilon}(\varepsilon H^2_T) = \varepsilon\cdot\operatorname{Fit}^{a-r}_{\mathbb{Z}_p[G]}(H^2_T).$$

Hence, the results follow from combining these observations with the isomorphism in (5.5). $\square$

Remark 5.3.2. In view of Theorem 3.3.1, the above isomorphism is equivalent to

$$I(\eta_{\mathcal{X}}) = \varepsilon_a\cdot\operatorname{Fit}^{a-r}_{\mathbb{Z}_p[G]}(H^2_T(\mathcal{O}_{F,S},\mathbb{Z}_p(1-j))). \tag{5.6}$$

Suppose $a=r$ (which is implied by the validity of Schneider's conjecture [65]). Let λ_j be the period-regulator isomorphism at j defined as in [18, §2.2]. Then $\mathcal{L}=\varepsilon\cdot\theta^*_{F/k,S,T}(j)^{-1}$ is conjecturally a characteristic element associated with the pair $(C^\varepsilon_{F,S,T}(j),\lambda_j)$ (see Conjecture 4.3.2). Let $\mathcal{X}=W^\varepsilon_j$ be the canonical basis of $\varepsilon Y_F(-j)$ fixed in §4.2.2. According to Proposition 4.3.3, the higher special element $\eta_{\mathcal{X}}$ in this setting coincides with the generalised Stark element $\eta^\varepsilon_{F/k,S,T}(j)$ in Definition 4.2.6 and the equality (5.6) recovers Conjecture 4.3.1 formulated by Burns, Kurihara and Sano in [18].

5.3.2 On the Galois Structure of Higher K-groups

In this section, we set $\mathbb{Z}'=\mathbb{Z}[1/2]$ and for each abelian group write M' in place of $\mathbb{Z}'\otimes_{\mathbb{Z}} M$. We also fix an integer m with $m>1$.

We fix an integer f with $f\not\equiv 2 \pmod 4$ and write F for the field generated by a primitive f-th root of unity in $\mathbb{C}$. We write Σ for the set of places of $\mathbb{Q}$ comprising ∞ and all prime divisors of f and set $\varepsilon^-_m := (1-(-1)^m\tau)/2$ where τ is the complex conjugation in $G=\operatorname{Gal}(F/\mathbb{Q})$.

We write $\epsilon_m(\zeta_f)$ for Beilinson's 'cyclotomic element' in $\mathbb{Q}\otimes_{\mathbb{Z}} K_{2m-1}(\mathcal{O}_F)$, as described by

Neukirch in [56, Part II, §1], and then set $c_F(m) := 2^{-1}(m-1)!f^{m-1}\cdot\epsilon_m(\zeta_f)$.

Theorem 5.3.3. The element $c_F(m)$ belongs to $\varepsilon_m^-\cdot K_{2m-1}(\mathcal{O}_F)'$ and there exists a canonical isomorphism of $\mathbb{Z}'[G]$-modules

$$\left(\frac{\varepsilon_m^-\cdot K_{2m-1}(\mathcal{O}_F)'}{\mathbb{Z}'[G]\cdot c_F(m)}\right)^\vee \cong \frac{\mathbb{Z}'[G]\varepsilon_m^-}{\varepsilon_m^-\cdot \mathrm{Fit}^0_{\mathbb{Z}'[G]}(K_{2m-2}(\mathcal{O}_{F,\Sigma})')}.$$

Proof. We fix an odd prime p and write S for $\Sigma\cup\{p\}$. Then the known validity of the Quillen-Lichtenbaum conjecture implies that there exists for both $k=1$ and $k=2$ a canonical Chern character isomorphism $\mathbb{Z}_p\otimes_{\mathbb{Z}} K_{2m-k}(\mathcal{O}_{F,\Sigma})\cong H^k(\mathcal{O}_{F,S},\mathbb{Z}_p(m))$.

In addition, in the case $k=1$ one has $K_{2m-k}(\mathcal{O}_{F,\Sigma}) = K_{2m-k}(\mathcal{O}_F)$ and, by Huber and Wildeshaus [43, Cor. 9.7], the image of $1\otimes c_F(m)$ under the Chern character isomorphism coincides with the cyclotomic element of Deligne and Soulé in $H^1(\mathcal{O}_{F,S},\mathbb{Z}_p(m))$.

Since this is true for all odd primes p it implies, in particular, that the element $c_F(m)$ belongs to $\varepsilon_m^-\cdot K_{2m-1}(\mathcal{O}_F)'$, as claimed. It is known, by the main result of [15], that $\varepsilon_m^-\theta^*_{F/\mathbb{Q},S,T}(1-m)$ is a characteristic element for the pair $(C^{\varepsilon_m^-}_{F/\mathbb{Q},S,T}(1-m),\lambda_{1-m})$. Observe that $\varepsilon_m^- H^1(\mathcal{O}_{F,S},\mathbb{Z}_p(m))$ is $\mathbb{Z}_p$-torsion free so the complex $C^{\varepsilon_m^-}_{F/\mathbb{Q},S,\emptyset}(1-m)$ is strictly admissible. Note that the rank of $\varepsilon_m^- Y_F(m-1)$ over $\mathbb{Z}_p[G]\varepsilon$ equals to 1. Now fix $\mathcal{X} = W^{\varepsilon_m^-}_{1-m}$ as defined in §4.2.2, Theorem 5.3.1 specialises to imply the following isomorphism

$$\left(\frac{\varepsilon_m^-\cdot H^1_T(\mathcal{O}_{F,S},\mathbb{Z}_p(m))}{\mathbb{Z}_p[G]\cdot \eta_{\mathcal{X}}}\right)^\vee \cong \frac{\mathbb{Z}_p[G]}{\mathrm{Fit}^0_{\mathbb{Z}_p[G]}(\varepsilon_m^-\cdot H^2_T(\mathcal{O}_{F,S},\mathbb{Z}_p(m)))}.$$

Hence it suffices to show that the higher special element $\eta_{\mathcal{X}}$ in this context coincides with Deligne-Soulé's cyclotomic element. This has been proven in [18, §5.1]. $\square$

5.3.3 Remarks on a result of El Boukhari

To describe the connection of Theorem 5.3.3 with the main result of El Boukhari in [31], we will prove the following lemma. At the outset, we let F/k be a finite abelian extension of group G and let S be a finite set of places of k containing the archimedean places, the p-adic places and the places that ramify in F.

Lemma 5.3.4. Fix a prime p and write H for the maximal subgroup of G of order prime to p. Let χ be any homomorphism $H \to \mathbb{Q}_p^{c\times}$ that satisfies the following condition

($*$) for each v in $S \setminus S_\infty(k)$ with both $v \nmid p$ and $p \mid \#I_v$ one has $\chi(H \cap I_v) \neq 1$,

where I_v is the inertia subgroup of v in G. Then, setting $R_\chi := \mathbb{Z}_p[G]^\chi$, one has

$$\mathrm{Fit}^0_{R_\chi}(K_{2m-2}(\mathcal{O}_{F,S})_p^\chi) = \mathrm{Fit}^0_{R_\chi}(K_{2m-2}(\mathcal{O}_F)_p^\chi) \cdot \prod_{v \in \Sigma \setminus S_\infty} (1 - \mathrm{Fr}_w^{-1} \cdot \ell_v^{m-1}) e_{I_v},$$

where w is a fixed place of F above v, Fr_w is its Frobenius automorphism in G/I_v, ℓ_v is the residue characteristic of v and e_{I_v} is the idempotent $(\#I_v)^{-1}\sum_{g \in I_v} g$.

Proof. First we recall that there is a canonical short exact sequence of R_χ-modules (due to Soulé [74])

$$0 \to K_{2m-2}(\mathcal{O}_F)_p^\chi \to K_{2m-2}(\mathcal{O}_{F,S})_p^\chi \to (\bigoplus_w K_{2m-3}(\kappa_w)_p)^\chi \to 0 \tag{5.7}$$

where w runs over non-archimedean places of F above S and κ_w denotes the corresponding residue field.

In addition, for any fixed prime p, the results of Quillen in [60] imply that $K_{2m-3}(\kappa_w)_p$ vanishes if w is p-adic and that if w is not p-adic, then $K_{2m-3}(\kappa_w)_p$ is naturally isomorphic to $H^1(\kappa_w, \mathbb{Z}_p(m-1))$ and hence that there is a natural short exact sequence of R_χ-modules

$$0 \to (\mathbb{Z}_p[G]e_{I_v})^\chi \xrightarrow{1-\mathrm{Fr}_w^{-1}\cdot\ell_v^{m-1}} (\mathbb{Z}_p[G]e_{I_v})^\chi \to (\bigoplus_{w|v} K_{2m-3}(\kappa_w)_p)^\chi \to 0. \tag{5.8}$$

Now, under the stated conditions on χ, all terms of the latter sequence vanish unless $\#I_v$ is prime to p in which case $(\mathbb{Z}_p[G]e_{I_v})^\chi$ is a direct summand of R_χ (hence projective over R_χ).

This observation shows that in all cases the R_χ-module $(\bigoplus_{w|v} K_{2m-3}(\kappa_w)_p)^\chi$ is cyclic and hence that

$$\begin{aligned}\mathrm{Fit}^0_{R_\chi}((\bigoplus_{w|v} K_{2m-3}(\kappa_w)_p)^\chi) =&\mathrm{Ann}_{R_\chi}((\bigoplus_{w|v} K_{2m-3}(\kappa_w)_p)^\chi)\\ =&R_\chi \cdot (1-\mathrm{Fr}_w^{-1}\cdot \ell_v^{m-1})e_{I_v}.\end{aligned}$$

The sequence (5.8) also implies that the observation of Cornacchia and Greither in [27, Lemma 3] can be applied to the sequence (5.7) to deduce that

$$\begin{aligned}&\mathrm{Fit}^0_{R_\chi}(K_{2m-2}(\mathcal{O}_{F,S})_p^\chi)\\ =&\mathrm{Fit}^0_{R_\chi}(K_{2m-2}(\mathcal{O}_F)_p^\chi)\cdot \prod_{v\in S\setminus S_\infty} \mathrm{Fit}^0_{R_\chi}((\bigoplus_{w|v} K_{2m-3}(\kappa_w)_p)^\chi)\\ =&\mathrm{Fit}^0_{R_\chi}(K_{2m-2}(\mathcal{O}_F)_p^\chi)\cdot \prod_{v\in S\setminus S_\infty} (1-\mathrm{Fr}_v^{-1}\cdot \ell_v^{m-1})e_{I_v},\end{aligned}$$

as claimed. □

Remark 5.3.5. If we return to the set-up of Theorem 5.3.3 (in particular $F=\mathbb{Q}(\mu_m)$ and $k=\mathbb{Q}$), the isomorphism in Theorem 5.3.3 directly implies an equality

$$\mathrm{Fit}^0_{\mathbb{Z}'[G]}\left(\left(\frac{\varepsilon_m^-\cdot K_{2m-1}(\mathcal{O}_F)'}{\mathbb{Z}'[G]\cdot c_F(m)}\right)^\vee\right) = \mathrm{Fit}^0_{\mathbb{Z}'[G]}(\varepsilon_m^-\cdot K_{2m-2}(\mathcal{O}_{F,\Sigma})').$$

Fix an odd prime p and write H for the maximal subgroup of G prime to p and P for the Sylow p-subgroup of G. If one first tensors the above equality with $\mathbb{Z}_p$, specialises to the case that m is odd, and also projects to the χ-component of the algebra $\mathbb{Z}_p[G]\varepsilon_m^-$ where χ is any character of H that satisfies the condition stated in Lemma 5.3.4, then we obtain the following equality

$$\begin{aligned}\mathrm{Fit}^0_{R_\chi}&\left(\left(\left(\frac{K_{2m-1}(\mathcal{O}_{F^+})\otimes_{\mathbb{Z}}\mathbb{Z}_p}{\mathbb{Z}_p[G]\cdot\eta_{\mathrm{cyc}}}\right)^{\chi}\right)^{\vee}\right)\\ &= \mathrm{Fit}^0_{R_\chi}((K_{2m-2}(\mathcal{O}_{F^+})\otimes_{\mathbb{Z}}\mathbb{Z}_p)^{\chi})\cdot\prod_{\ell\in\Sigma\setminus\{\infty\}}(1-\mathrm{Fr}_\ell^{-1}\cdot\ell^{m-1})e_{I_\ell}, \quad (5.9)\end{aligned}$$

where F^+ is the maximal real subfield of F and η_{cyc} denotes the Deligne-Soulé cyclotomic element.

The above equality in effect constitutes a ‘corrected form’ of the main result of El Boukhari claimed in [31]. To be specific, the factor ‘$Q(0)$’ that occurs in the statement of [31, §1, Th. 6.5] was inaccurately described in the proof of [31, Lem. 5.3]. In particular, since every ℓ that occurred in the definition of ‘$Q(0)$’ divides the conductor of F and hence ramifies, using the description in loc. cit. would lead to having $Q(0)\equiv 1$.

5.4 Dirichlet L-functions at $s=1$

In this section, we study the leading term of the equivariant L-function at the value $s=1$ by the theory of higher special elements that we developed in §3. By combining the main theorem in this section (Theorem 5.4.1) with classical conjectures regarding the values of the p-adic L-series at $s=1$, we formulate a new conjecture that is an analogue of the classical Brumer’s conjecture. Conjectures of this kind in a different setting are previously formulated by Castillo and Jones in [26] and by Solomon in [72].

At the outset we recall that F/k is an abelian extension of number fields and we fix an odd prime p. We assume that S also contains the p-adic places of k. We write $C(1) := R\Gamma_c(\mathcal{O}_{k,S}, \mathbb{Z}_p(1)_F)$ of $D^{\mathrm{a}}(\mathbb{Z}_p[G])$ as defined in §2.4.2 and $Y_F := Y_F(1)$ as defined in §4.2.2. We also write A_F for the Sylow p-subgroup of the ideal class group of F and $M_S(F)$ for the maximal pro-p abelian extension of F unramified outside S. Also for each p-adic place w of F we write U_w for the (pro-p) group of principal units in the completion of F at w.

5.4.1 Annihilating the p-part of ideal class groups

To formulate the main theorem in this section, we shall make use of the technique developed in §2.3. Suppose we are given a homomorphism of $\mathbb{Z}_p[G]$-modules of the form

$$\phi : Y_F \to \prod_{w|p} U_w.$$

Proposition 2.4.10 gives an exact triangle in $D(\mathbb{Z}_p[G])$

$$Y_F[-1] \oplus Y_F[-2] \to C(1) \to C_\phi(1) \to Y_F[0] \oplus Y_F[-1] \tag{5.10}$$

for a complex $C_\phi(1)$ in $D^{\mathrm{a}}(\mathbb{Z}_p[G])$.

We set $T_{\mathbf{1}} := \sum_{g\in G} g$ and write $e_{\mathbf{1}}$ for the idempotent $|G|^{-1}T_{\mathbf{1}}$ of $\mathbb{Q}_p[G]$. We set $\mathfrak{B} := \mathbb{Z}_p[G](1-e_{\mathbf{1}})$ and also $B := \mathbb{Q}_p[G](1-e_{\mathbf{1}})$. We note that the natural short exact sequence of $\mathbb{Z}_p[G]$-modules

$$0 \to (T_{\mathbf{1}}) \to \mathbb{Z}_p[G] \xrightarrow{1\mapsto 1-e_{\mathbf{1}}} \mathfrak{B} \to 0 \tag{5.11}$$

implies that for a $\mathbb{Z}_p[G]$-module N there is a natural action of $\mathfrak{B}$ on the quotient $N/H^0(G,N)$. For any $\mathbb{Z}_p[G]$-module M and perfect complex C of $\mathbb{Z}_p[G]$-modules , we set $M_0 := \mathfrak{B} \otimes_{\mathbb{Z}_p[G]} M$

and $C_0 := \mathfrak{B} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C$.

In the sequel we write e_* for the sum of all primitive idempotents e in B with the property that $e(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(C_\phi(1))_0)$, and hence also $e(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(C_\phi(1))_0)$, vanishes. Now we state the main theorem of this section.

Theorem 5.4.1. The following claims are valid.

(i) One has $e_*(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} Y_F) = e_*(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(C(1)_0))$ and the map

$$\phi_{(e_*)} : e_*(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(C(1)_0)) \to e_*(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(C(1)_0))$$

induced by the composite of ϕ and the reciprocity map $\kappa_p : \prod_{w|p} U_w \to \mathrm{Gal}(M_S(F)/F)$ is bijective.

(ii) Let $\mathcal{L}$ be a characteristic element of $C(1)_0$ with respect to an isomorphism ψ of $B_{\mathbb{C}_p}$-modules $\mathbb{C}_p \otimes_{\mathbb{Z}_p} H^1(C(1)_0) \to \mathbb{C}_p \otimes_{\mathbb{Z}_p} H^2(C(1)_0)$ that agrees with $\mathbb{C}_p \otimes_{\mathbb{Q}_p} \phi_{(e_*)}$ on $e_*(\mathbb{C}_p \otimes_{\mathbb{Z}_p} H^1(C(1)_0))$.

Then the product $e_* \cdot \mathcal{L}^{-1}$ belongs to $\mathfrak{B}$ and annihilates the module $A_F/H^0(G, A_F)$.

Proof. By Lemma 2.3.5, the complex $C_\phi(1)_0 := \mathfrak{B} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C_\phi(1)$ belongs to $D^{\mathrm{a}}(\mathfrak{B})$. Since $H^3(C_\phi(1)) = H^3(C(1))$ identifies with $\mathbb{Z}_p$, the sequence (5.11) implies that the group $H^3(C_\phi(1)_0)$ is finite. Hence the B-modules $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(C_\phi(1)_0)$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(C_\phi(1)_0)$, and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(C(1)_0)$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(C(1)_0)$, are isomorphic (as a consequence of the respective conditions (ad_2) and (ad_3)). Then claim (i) follows directly from the definition of e_* and the long exact sequence of cohomology of the triangle (5.10).

To prove claim (ii), first we note that since $H^3(C_\phi(1)) = \mathbb{Z}_p$, one has $e_0 = 1 - e_{\mathbf{1}}$ and $\mathrm{Fit}^0_{\mathbb{Z}_p[G]}(H^3(C_\phi(1)))$ identifies with the augmentation ideal $I_p(G) := \mathbb{Z}_p[G] \cap B$ of $\mathbb{Z}_p[G]$.

Thus, for some x in $I_p(G) \cap B^\times$, Proposition 2.3.8 implies the existence of a complex

$C_x = C_\phi(1)_x$ in $D^{\mathrm{s}}(\mathfrak{B})$ with the property that $H^2(C_x)$ contains $H^2(C_\phi(1)_0)$ and $x \cdot \mathrm{Det}_{\mathfrak{B}}(C_x) = e_0 \cdot \mathrm{Det}_{\mathbb{Z}_p[G]}(C_\phi(1)) = \mathrm{Det}_{\mathfrak{B}}(C_\phi(1)_0)$.

In addition, by the definition of e_*, one has $e_*(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(C_\phi(1))_0)$ and hence also $e_*(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(C_\phi(1))_0)$, vanishes. The complex $Be_* \otimes_{\mathfrak{B}} C_x$ is therefore acyclic, one can thus identify $e_* \cdot \mathrm{Det}_{\mathfrak{B}}(C_x)^{-1}$ as a sublattice of B. With respect to this identification, the equality $e_* = e_* e_0$ implies that

$$\begin{aligned} &e_* \cdot \mathrm{Det}_{\mathbb{Z}_p[G]}(C_\phi(1))^{-1} = e_* \cdot \mathrm{Det}_{\mathfrak{B}}(C_\phi(1)_0)^{-1} = x e_* \cdot \mathrm{Det}_{\mathfrak{B}}(C_x)^{-1} \\ &\subseteq x \cdot \mathrm{Ann}_{\mathfrak{B}}(H^2(C_x)) \subseteq x \cdot \mathrm{Ann}_{\mathfrak{B}}(H^2(C_\phi(1)_0)) \subseteq \mathrm{Ann}_{\mathfrak{B}}(H^0(G, \mathfrak{B} \otimes_{\mathbb{Z}_p} H^2(C_\phi(1)))). \end{aligned} \tag{5.12}$$

Here the first inclusion follows from the argument of Proposition 3.2.9 (ii), the second is by the inclusion $H^2(C_\phi(1)_0) \subseteq H^2(C_x)$. To show the third inclusion, we first write down the low degree terms of the spectral sequence (see [79, §5])

$$H^a(G, \mathfrak{B} \otimes_{\mathbb{Z}_p} H^b(C_\phi(1))) \Longrightarrow H^{a+b}(C_\phi(1)_0)$$

to obtain an exact sequence

$$H^2(C_\phi(1)_0) \to H^0(G, \mathfrak{B} \otimes_{\mathbb{Z}_p} H^2(C_\phi(1))) \to H^2(G, \mathfrak{B} \otimes_{\mathbb{Z}_p} H^1(C_\phi(1))),$$

then we combine it with the short exact sequence (5.11) to show that $H^2(G, \mathfrak{B} \otimes_{\mathbb{Z}_p} H^1(C_\phi(1)))$ is isomorphic to $H^3(G, (T_{\mathbf{1}}) \otimes_{\mathbb{Z}_p} H^1(C_\phi(1)))$ and hence is annihilated by x.

Next, we note that the functor $- \otimes_{\mathbb{Z}_p} H^2(C_\phi(1))$ preserves the exactness of the sequence (5.11) and hence, upon taking G-invariants, gives an exact sequence

$$0 \to H^0(G, (T_{\mathbf{1}} \otimes_{\mathbb{Z}_p} H^2(C_\phi(1)))) \to H^0(G, \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} H^2(C_\phi(1))) \to H^0(G, \mathfrak{B} \otimes_{\mathbb{Z}_p} H^2(C_\phi(1))).$$

Since the third module in this sequence identifies with $H^2(C_\phi(1))$ in such a way that the image of the second arrow is equal to $H^0(G, H^2(C_\phi(1)))$, we deduce from (5.12) that any element in $e_* \cdot \mathrm{Det}_{\mathfrak{B}}(C_\phi(1)_0)^{-1}$ belongs to $\mathfrak{B}$ and annihilates $H^2(C_\phi(1))/H^0(G, H^2(C_\phi(1)))$.

By the construction of Proposition 2.4.10, A_F is isomorphic to a quotient of $H^2(C_\phi(1))$. This implies that $A_F/H^0(G, A_F)$ is isomorphic to a quotient of $H^2(C_\phi(1))/H^0(G, H^2(C_\phi(1)))$. Hence, by the properties stated in Proposition 2.3.8, we deduce that any such element annihilates $A_F/H^0(G, A_F)$.

It is thus enough to show that for any element $\mathcal{L}$ as in claim (ii) one has $e_* \cdot \mathcal{L}^{-1} \in e_* \cdot \mathrm{Det}_{\mathfrak{B}}(C_\phi(1)_0)^{-1}$.

To do this we combine the standard identification $\mathrm{Det}_{\mathbb{Z}_p[G]}(Y_F[-1] \oplus Y_F[-2]) = (\mathbb{Z}_p[G], 0)$ with the exact triangle in $D(\mathfrak{B})$ that is obtained by applying $\mathfrak{B} \otimes_{\mathbb{Z}_p[G]} -$ to (5.10) to obtain an identification ϑ_ψ of $\mathfrak{B}$-modules $\mathrm{Det}_{\mathfrak{B}}(C_\phi(1)_0) \to \mathrm{Det}_{\mathfrak{B}}(C(1)_0)$ that has the following property: the restriction of ϑ_ψ to $e_* \cdot \mathrm{Det}_{\mathfrak{B}}(C(1)_0)$ coincides with the composite $e_* \cdot \mathrm{Det}_{\mathfrak{B}}(C(1)_0) = e_* \cdot \mathrm{Det}_{\mathfrak{B}}(C_\phi(1)_0) \subset e_* \cdot B$, where the inclusion is as used in (5.12).

This implies, in particular, that $e_* \cdot \vartheta_\psi^{-1}(\mathcal{L}) = e_* \cdot \mathcal{L}$ belongs to $e_* \cdot \mathrm{Det}_{\mathfrak{B}}(C_\phi(1)_0)$, as required. $\square$

5.4.2 Connections with p-adic L-series

To put Theorem 5.4.1 in context, we formulate in this section a new and explicit conjecture regarding the leading term of the p-adic L-series at $s = 1$. For an abelian extension of totally real fields F/k of group G, we define

$$\mathcal{L}_{F/k,S,p}(s) := \sum\nolimits_{\rho \in \widehat{G}_p \setminus \{\mathbf{1}\}} L_{p,S}(s, \rho) \cdot e_\rho$$

where $L_{p,S}(s, \rho)$ is the S-truncated Kubota-Leopoldt p-adic L-series of non-trivial homomorphisms ρ in $\widehat{G}_p := \mathrm{Hom}(G, \mathbb{Q}_p^{c\times})$.

Conjecture 5.4.2. Let F/k be an abelian extension of totally real number fields of Galois group G. Then the element $\mathcal{L}_{F/k,S,p}(1)$ belongs to $\mathbb{Z}_p[G](1-e_{\mathbf{1}})$ and annihilates the module $A_F/H^0(G,A_F)$.

In the sequel, we derive evidence of the above conjecture by describing its connection with Theorem 5.4.2 and existing conjectures regarding p-adic L-series in the literature.

To start with, we write Σ_F for the set of field embeddings $F \to \mathbb{C}$, define

$$H_F := \bigoplus_{\Sigma_F} \mathbb{Z} \cdot (2\pi i)$$

for the $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \times G$-module upon which $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts diagonally via post-composition on Σ_F and on $2\pi i$ in the natural way, and G acts via pre-composition on Σ_F.

We use the canonical composite isomorphism of $\mathbb{C}_p[G]$-modules

$$\varpi : \mathbb{C}_p \otimes_{\mathbb{Q}} F \cong \prod_{\Sigma_F} \mathbb{C}_p \cong \mathbb{C}_p \otimes_{\mathbb{Z}} H_F,$$

where the first isomorphism sends each element $z \otimes f$ to $(z\sigma(f))_\sigma$ and the second is induced by the identification $\mathbb{C}_p = \mathbb{C}_p \otimes_{\mathbb{Z}} \mathbb{Z}(2\pi i)$.

We also fix a topological generator ξ of $\mathbb{Z}_p(1)$ and note that the assignment $\xi \mapsto 2\pi i$ identifies Y_F with the submodule of $\mathbb{Z}_p \otimes_{\mathbb{Z}} H_F$ upon which $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts trivially.

We then consider the composite homomorphisms of $\mathbb{C}_p[G]$-modules

$$\begin{aligned}\lambda_F^p : (\mathbb{C}_p \otimes_{\mathbb{Z}_p} Y_F)_0 \subset (\mathbb{C}_p \otimes_{\mathbb{Z}} H_F)_0 &\xrightarrow{\varpi^{-1}} (\mathbb{C}_p \otimes_{\mathbb{Q}} F)_0 = (\bigoplus_{w|p} \mathbb{C}_p \otimes_{\mathbb{Q}_p} F_w)_0 \\ &\xrightarrow{(\mathbb{C}_p \otimes_{\mathbb{Q}_p} \exp_F^p)_0} (\bigoplus_{w|p} \mathbb{C}_p \otimes_{\mathbb{Z}_p} U_w)_0 \xrightarrow{\kappa} (\mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathrm{Gal}(M_S(F)/F))_0 = (\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^2(C(1)))_0\end{aligned}$$

and

$$\lambda_F^{\infty,p} : (\mathbb{C}_p \otimes_{\mathbb{Z}} Y_F)_0 \xrightarrow{\mathbb{C}_p \otimes_{\mathbb{R}} \exp_F^\infty} (\mathbb{C}_p \otimes_{\mathbb{Z}} \mathcal{O}_F^\times)_0 \xrightarrow{1 \otimes \Delta_p} (\bigoplus_{w|p} \mathbb{C}_p \otimes_{\mathbb{Z}_p} U_w)_0$$
$$\xrightarrow{\varpi \circ (\mathbb{C}_p \otimes_{\mathbb{Q}_p} \exp_F^p)_0^{-1}} (\mathbb{C}_p \otimes_{\mathbb{Z}} H_F)_0 \xrightarrow{\kappa'} (\mathbb{C}_p \otimes_{\mathbb{Z}} Y_F)_0.$$

Here $\exp_F^p$ denotes the product (over w) of the p-adic exponential maps $F_w \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} U_w$ and κ is induced by the global reciprocity map. In addition, we write $\exp_F^\infty$ for the inverse of the Dirichlet regulator isomorphism $(\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_F^\times)_0 \cong (\mathbb{R} \otimes_{\mathbb{Z}} Y_F)_0$ in (4.6), Δ_p for the natural diagonal map $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F^\times \to \prod_{w|p} U_w$ and κ' for the map induced by sending each element $(2\pi i) \cdot \sigma$, with $\sigma \in \Sigma_F$, to the place that corresponds to σ.

To explain the significance of the maps λ_F^p and $\lambda_F^{\infty,p}$ we write

$$\vartheta^{\mathrm{BK}} : \mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathrm{Det}_{\mathbb{Z}_p[G]}(C(1)) \cong (\mathbb{C}_p[G], 0)$$

for the isomorphism of $B_{\mathbb{C}_p}$-modules that occurs in the Tamagawa Number Conjecture of Bloch and Kato for the pair $(h^0(\mathrm{Spec}(F)(1), \mathbb{Z}_p[G])$, as described, for example in the course of the proof of [25, Th. 8.1]. We remark that the definition of ϑ^{BK} is closely related to the period-regulator isomorphism for $j = 1$ defined in §4.2.2.

We then fix an element $\mathcal{L}^{\mathrm{BK}}$ of $B_{\mathbb{C}_p}$ such that $\vartheta^{\mathrm{BK}}(\mathrm{Det}_{\mathfrak{B}}(C(1)_0)) = (\mathfrak{B} \cdot \mathcal{L}^{\mathrm{BK}}, 0)$.

Lemma 5.4.3. Assume that the map $e_* \cdot \lambda_F^p$ is bijective. Then the map $e_* \cdot \lambda_F^{\infty,p}$ is also bijective and there exists an element $\mathcal{L}$ as in Proposition 5.4.1 for which one has

$$e_* \cdot \mathcal{L}^{-1} = \mathcal{R}_F^{\infty,p} \cdot \mathcal{R}(\phi) \cdot (\mathcal{L}^{\mathrm{BK}})^{-1}$$

with $\mathcal{R}_F^{\infty,p} := \det_{e_* B_{\mathbb{C}_p}}(e_* \cdot \lambda_F^{\infty,p})^{-1}$ and $\mathcal{R}(\phi) := \det_{e_* B_{\mathbb{C}_p}}(e_*(\mathbb{C}_p \otimes_{\mathbb{Z}_p} \phi) \circ (e_* \cdot \lambda_F^p)^{-1})$

Proof. The definition of e_* (given just before the statement of Theorem 5.4.1) implies that $e_*(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(C_\phi(1))_0)$ vanishes and this in turn implies that the map $e_*(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Delta_p)$ is injective.

Thus, if $e_* \cdot \lambda_F^p$ is bijective, then the bijectivity of $e_* \cdot \lambda_F^{\infty,p}$ follows from the fact that $\mathrm{Im}(\Delta_p)$ is equal to the kernel of the reciprocity map $\prod_{w|p} U_w \to \mathrm{Gal}(M_S(F)/F)$.

In addition, if these maps are bijective, then by unwinding the explicit definition of ϑ^{BK} one finds that

$$e_* \cdot \vartheta^{\mathrm{BK}} = (\mathcal{R}_F^{\infty,p})^{-1} \cdot \vartheta_{\phi_{(e_*)}} = (\mathcal{R}_F^{\infty,p})^{-1} \cdot \mathcal{R}(\phi)^{-1} \cdot e_* \vartheta_\psi.$$

where ϑ_ψ is as defined at the end of the proof of Theorem 5.4.1. This implies the second claim since $(\vartheta_\psi \circ (\vartheta^{\mathrm{BK}})^{-1})(\mathcal{L}^{\mathrm{BK}})$ is a characteristic element of $C_\phi(1)_0$ with respect to ϑ_ψ. □

Theorem 5.4.4. Let F/k be an abelian extension of totally real number fields of Galois group G. Fix p to be an odd prime. If the following conjectures are valid:

(i) the Equivariant Tamagawa Number Conjecture for the pair $(h^0(\mathrm{Spec}(F))(1), \mathbb{Z}_p[G])$,

(ii) the 'p-adic Stark conjecture of F at $s = 1$' for F/k (as formulated in [24, Conj. 5.2]),

then Conjecture 5.4.2 is valid.

Proof. First we note that according to [24, Rem. 5.4], the validity of (ii) implies the validity of the Leopoldt's Conjecture of F at p. Therefore, we have that $e_* \cdot \lambda_F^p$ is bijective. In this case one has $e_* = 1 - e_{\mathbf{1}}$ and the module Y_F vanishes so $\mathcal{R}(\phi) = 1 - e_{\mathbf{1}}$. The validity of (i) implies that one can take $(\mathcal{L}^{\mathrm{BK}})^{-1}$ to be $\theta^*_{F/k,S}(1)$. Finally the validity of (ii) imply, according to [24, Rem. 5.3] and also [25, Rem. 7.2], that $\mathcal{R}_F^{\infty,p} \cdot (\mathcal{L}^{\mathrm{BK}})^{-1}$ can be interpreted in terms of the values at $s = 1$ of the S-truncated Kubota-Leopoldt L-series $L_{p,S}(s, \rho)$. Combining these observations with Lemma 5.4.3, the result follows from Theorem 5.4.1. □

Remark 5.4.5. If F is a totally real abelian extension of $\mathbb{Q}$, then the known validity of the Equivariant Tamagawa Number Conjecture for $(h^0(\mathrm{Spec}(F))(1), \mathbb{Z}_p[G])$ (see [15]) and of the p-adic Stark conjecture at $s = 1$ for homomorphisms in $\widehat{G}_p$ (see [42]) implies Conjecture 5.4.2 is valid unconditionally for F. In the special case that G is cyclic, weaker results of this form have previously been obtained by Oriat in [58].

Remark 5.4.6. In a forthcoming work of the author with David Burns and Takamichi Sano, by making use of the previous results of Burns and Macias Castillo [21], we prove that Conjecture 5.4.2 is valid if either the p-adic μ-invariant of F vanishes or p does not divide $[F : k]$.

The author was later informed by Henri Johnston that the p-adic μ-invariant of F vanishes for finite Galois p-extensions of finite abelian extensions of $\mathbb{Q}$ (see, for example, [42, Rem. 8.5]). One can use [42, Th. 10.5] to come up with certain relative cyclic totally real extensions F/k for which the Leopoldt's Conjecture for F at p implies the p-adic Stark conjecture at $s = 1$ for F/k. Hence, in this case, we have that the Leopoldt's Conjecture for F at p implies Conjecture 5.4.2 when p is odd.

Chapter 6

Higher Rank Iwasawa Theory

In this chapter, we develop the Iwasawa theory of the generalised Stark elements of totally real fields. The main conjecture in this context was first formulated by Burns, Kurihara and Sano in [17]. Their conjecture predicts the existence of an Iwasawa theoretical 'zeta element' that (solely) concerns the values at $s = 0$ of derivatives of Dirichlet L-functions. However our main result in this chapter suggests that, subject to the validity of the generalised Kummer congruences in §4.4, such an interpolation property actually implies an interpolation property at arbitrary even integers.

At the outset we fix an odd prime number p and a finite abelian extension of totally real number fields L/K. We will adopt the notations from §4.2. In particular, for any CM extension L'/K, we denote by c the complex conjugation in $\mathrm{Gal}(L'/K)$ and we write $\varepsilon^+ = (1+c)/2$ to be the idempotent in $\mathbb{Z}_p[\mathrm{Gal}(L'/K)]$.

6.1 The higher rank Iwasawa main conjecture

6.1.1 Iwasawa theoretical set-up

We set K_∞ to be the cyclotomic $\mathbb{Z}_p$-extension over K and for simplicity we assume $L \cap K_\infty = K$. Then $L_\infty := LK_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension over L.

We write $\mathcal{G} := \mathrm{Gal}(L_\infty/K) = \Delta \times \Gamma$ with Δ finite and Γ topologically isomorphic to $\mathbb{Z}_p$. We also identify $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(L_\infty/L)$ with Δ and Γ in the obvious way. We write Λ for the p-adic Iwasawa algebra $\varprojlim_U \mathbb{Z}_p[\mathcal{G}/U]$, where U runs over all open subgroups of $\mathcal{G}$ and the transition morphisms are the natural projection maps.

Fix a finite set of places S of K containing all archimedean places S_∞, all those that ramify in L_∞/K and (hence also) all p-adic places. We set $r = |S_\infty|$ (which is equal to $[K : \mathbb{Q}]$ in this case). We also let T be an auxiliary finite set of places of K that is disjoint from S. Recall that for every finite extension M/K we have defined in §4.2.1 a T-modified version of the étale cohomology $C_{M,S,T}(j)$ of the Tate module $\mathbb{Z}_p(1-j)$. To be consistent with the gradings used in [17], we will use the complex $C_{M,S,T}(j)[1]$ instead. Then, by passing to the limit over finite subextensions E/K of L_∞/K, we obtain a perfect complex

$$C_{L_\infty,S,T}(j) := \varprojlim_M C_{M,S,T}(j)[1]$$

of Λ-modules.

We fix an isomorphism of fields $\mathbb{C} \cong \mathbb{C}_p$ and use it to identify $\widehat{\Delta}$ with $\mathrm{Hom}_{\mathbb{Z}}(\Delta, \mathbb{Q}_p^c)$. For each χ in $\widehat{\Delta}$ we set

$$L_\chi := L^{\ker(\chi)},\ L_{\chi,\infty} := L_\chi K_\infty \ \text{ and }\ \mathcal{G}_\chi := \mathrm{Gal}(L_{\chi,\infty}/K). \tag{6.1}$$

For a finite order character ψ in $\mathrm{Hom}(\mathcal{G}_\chi, \mathbb{C}^\times)$ we then set

$$L_\psi := (L_{\chi,\infty})^{\ker(\psi)} \quad \text{and} \quad G_\psi := \mathrm{Gal}(L_\psi/K). \tag{6.2}$$

We do not distinguish between ψ and its image in $\widehat{G_\psi}$ and write e_ψ for the associated idempotent in $\mathbb{C}[G_\psi]$.

6.1.2 Statement of the conjecture

To describe the interpolation property of the Iwasawa theoretical zeta element (and hence formulate the main conjecture), we first extend the constructions of [17] to define a canonical 'trivialisation' homomorphism

$$\lambda_j^\psi : \mathrm{Det}_\Lambda(C_{L_\infty,S,T}(j)) \to \mathbb{C}_p \tag{6.3}$$

as follows. Fix a character $\chi \in \widehat{\Delta}$. For each pair $\psi \in \widehat{\mathcal{G}_\chi}$ and integer j such that

$$\dim_{\mathbb{C}_p} e_\psi \mathbb{C}_p H^1(\mathcal{O}_{L_\psi,S}, \mathbb{Q}_p(1-j)) = \dim_{\mathbb{C}_p} e_\psi \mathbb{C}_p Y_{L_\psi}(-j),$$

we specify the homomorphism λ_ψ^j in (6.3) to be equal to the following composite

$$\begin{aligned}
&\mathrm{Det}_\Lambda(C_{K_\infty,S,T}(j))\\
\to\ &\mathrm{Det}_{\mathbb{Z}_p[G_\psi]}(C_{L_\psi,S,T}(j)[1])\\
\to\ &\mathrm{Det}_{e_\psi\mathbb{C}_p[G_\psi]}(e_\psi\mathbb{C}_pC_{L_\psi,S,T}(j)[1])\\
\to\ &\mathrm{Det}_{e_\psi\mathbb{C}_p[G_\psi]}(e_\psi\mathbb{C}_pH^1(C_{L_\psi,S,T}(j)))\otimes\mathrm{Det}^{-1}_{e_\psi\mathbb{C}_p[G_\psi]}(e_\psi\mathbb{C}_pH^2(C_{L_\psi,S,T}(j)))\\
\to\ &\mathrm{Det}_{e_\psi\mathbb{C}_p[G_\psi]}(e_\psi\mathbb{C}_pH^2(C_{L_\psi,S,T}(j)))\otimes\mathrm{Det}^{-1}_{e_\psi\mathbb{C}_p[G_\psi]}(e_\psi\mathbb{C}_pH^2(C_{L_\psi,S,T}(j)))\\
\to\ &e_\psi\mathbb{C}_p[G_\psi]\\
\to\ &\mathbb{C}_p.
\end{aligned}$$

The maps involved in this composite are as follows: the first is induced by the standard descent isomorphism $\mathbb{Z}_p[G_\psi]\otimes^{\mathbb{L}}_\Lambda C_{K_\infty,S,T}(j)\cong C_{L_\psi,S,T}(j)[1]$ in $D^{\mathrm{p}}(\mathbb{Z}_p[G_\psi])$ and the behaviour of determinants under the ring extension $\mathbb{Z}_p[G_\psi]\otimes^{\mathbb{L}}_\Lambda-$; the second is induced by the behaviour of determinants under the ring extension $e_\psi(\mathbb{C}_p[G_\psi]\otimes_{\mathbb{Z}_p[G_\psi]}-)$; the third results from the canonical 'passage to cohomology' isomorphism of the determinant functor over the semisimple algebra $e_\psi\mathbb{C}_p[G_\psi]$ and the fact that the complex $e_\psi\mathbb{C}_pC_{L_\psi,S,T}(j)$ is acyclic outside degrees one and two; the fourth is equal to $e_\psi(\lambda_j)\otimes\mathrm{id}$ where the period-regulator isomorphism λ_j is as defined in §4.2.2; the fifth is the natural evaluation map; the sixth is induced by ψ.

Remark 6.1.1. In the case of $j=0$ the above composite agrees with the map that is defined in [17, §2.4].

We can now recall the central conjecture of [17]. (To help readers, we also recall that we denote the order vanishing of $L_{k,S,T}(\psi,z)$ at $z=0$ by $r_{\psi,S}(0)$)

Conjecture 6.1.2. (Burns, Kurihara and Sano) There exists a Λ-basis $\mathcal{L}_0$ of $\mathrm{Det}_\Lambda(C_{L_\infty,S,T}(0))$ with the following property: for every χ in $\widehat{\Delta}$ and every ψ in $\widehat{\mathcal{G}_\chi}$ with $r_{\psi,S}(0)=r$ one has

$\lambda_0^{\psi}(\mathcal{L}_0) = L^{(r)}_{K,S,T}(\psi^{-1}, 0)$.

Remark 6.1.3.

(i) This conjecture constitutes a natural main conjecture of (equivariant) Iwasawa theory in cases in which the relevant cohomology module can have strictly positive rank. The conjecture was used in [17] to study and develop strategies in proving an important special case of the Equivariant Tamagawa Number Conjecture.

(ii) In [17], the authors formulated a conjecture for any abelian extension of number fields. In this thesis, we only recall a special (but interesting) case of their conjecture. Under the more general set-up, if $r = 0$ (for example, when L is CM and K is totally real), the element $\mathcal{L}_0$ is the p-adic L-function in the classical main conjecture of Iwasawa. One may therefore regard $\mathcal{L}_0$ in Conjecture 6.1.2 as a higher rank analogue of the p-adic L-function.

We end this section by recalling a result concerning the validity of Conjecture 6.1.2.

Theorem 6.1.4. Conjecture 6.1.2 is valid if $K = \mathbb{Q}$.

Proof. This is a direct consequence of the classical Iwasawa Main Conjecture proven by Mazur and Wiles in [53] (see [15] and [34] for a derivation). □

In the sequel, we say that the higher rank Iwasawa Main Conjecture is valid for (L_∞/K) if Conjecture 6.1.2 is valid.

6.1.3 Relation with Rubin-Stark elements

In this section, we make a remark on the relation between the element $\mathcal{L}_0$ predicted in the statement of Conjecture 6.1.2 and the Rubin-Stark elements defined in §4.1. To do this, we recall a projection morphism defined in [17]. Recall that we have set $r := |S_\infty|$ to be the number of infinite places of K. For any finite abelian extension E/K of totally real number fields with Galois group G, we set e_r to be the sum of primitive idempotents e_χ such that $\chi \in \widehat{G}$ satisfies $r_{\chi,S}(0) = r$. Define the composition

$$\begin{aligned}
\pi^{S_\infty}_{E/K,S,T} : \mathrm{Det}_{\mathbb{Z}_p[G]}(C_{E,S,T}(0)[1]) &\xrightarrow{e_r\mathbb{C}_p\otimes} e_r\mathbb{C}_p \,\mathrm{Det}_{\mathbb{Z}_p[G]}(C_{E,S,T}(0)) \\
&\xrightarrow{\sim} e_r(\mathbb{C}_p \bigwedge^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{E,S}, \mathbb{Z}_p(1)) \otimes_{\mathbb{C}_p[G]} \mathbb{C}_p \bigwedge^r_{\mathbb{Z}_p[G]} Y_E(0)) \\
&\xrightarrow{\sim} e_r\mathbb{C}_p \bigwedge^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{E,S}, \mathbb{Z}_p(1)) \\
&\subset \mathbb{C}_p \bigwedge^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{E,S}, \mathbb{Z}_p(1))
\end{aligned}$$

where the first map is a natural ring extension morphism, the second map is the 'passage to cohomology' property of determinant functor, the third map is a (non-canonical) isomorphism induced by setting $x \otimes w_1 \wedge ... \wedge w_r \mapsto x$ where $\{w_i\}_{1\le i\le r}$ is the canonical basis of $Y_E(0)$ fixed in §4.2.3. Now we set

$$\pi^{S_\infty}_{L_{\chi,\infty}/K,S,T} := \varprojlim_n \pi^{S_\infty}_{L_{\chi,n}/K,S,T}$$

where $L_{\chi,n}$ denotes the 'n-th layer' of the cyclotomic $\mathbb{Z}_p$-extension $L_{\chi,\infty}/L$.

For a rational prime p, we say that the 'p-component' of Conjecture 4.1.4 is valid for the data $(L/K, S, T, V)$ if one has

$$\eta^V_{L/K,S,T} \in \bigcap^{|V|}_{\mathbb{Z}_p[G]} (\mathcal{O}^\times_{L,S,T} \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

Proposition 6.1.5. If Conjecture 6.1.2 is valid for (L_∞/K), then for every $\chi \in \widehat{\Delta}$, the '$p$-component' of the Rubin-Stark conjecture (Conjecture 4.1.4) is valid for the data $(L_{\chi,n}/K, S, T, S_\infty)$

for every n. Moreover, one has that

$$\pi^{S_\infty}_{L_{\chi,\infty}/K,S,T}(\mathcal{L}_0) = \varprojlim_n \eta^{S_\infty}_{L_{\chi,n}/K,S,T}.$$

Proof. These were proven in [17, Th. 3.4] and [17, Th. 3.5 (ii)] respectively. □

6.2 Interpolation properties of higher p-adic L-functions

In this section, we prove an interpolation formula of the zeta element in the statement of the higher rank main conjecture (Conjecture 6.1.2). We set L_p^+ to be the maximal totally real subfield of $L(\mu_p)$ and recall that $L^+_{p,\infty} = L_p^+ K_\infty$ denotes its cyclotomic $\mathbb{Z}_p$-extension. We are going to study the zeta element that arises from Conjecture 6.1.2 for the extension $L^+_{p,\infty}/K$. In the sequel, we adopt the notations from §6.1.1 with L_p^+ in place of L. In particular, we set $\Delta := \mathrm{Gal}(L_p^+/K)$ and $\mathcal{G} := \mathrm{Gal}(L^+_{p,\infty}/K)$. Note that we are going to fix a finite set T of places of K such that

$$\begin{cases} T \text{ and } S \text{ are disjoint}, \\ \varepsilon^+ H^1_T(\mathcal{O}_{L_{p^n},S}, \mathbb{Z}_p(1-j)) \text{ is } \mathbb{Z}_p\text{-torsion-free for all } n \geq 1 \text{ and even integer } j. \end{cases} \tag{6.4}$$

We remark that the second hypothesis is very mild according to Lemma 4.2.2.

We now recall the 'twisting morphisms' between perfect complexes of Λ-modules that appear in the work of Fukaya and Kato [36]. To do this, we set $\Lambda' := \varprojlim_U \mathbb{Z}_p[\mathrm{Gal}(L_{p,\infty}/K)/U]$ where U runs over all open subgroups of $\mathrm{Gal}(L_{p,\infty}/K)$. For any integer j, we define the ring automorphism $\mathrm{tw}_j : \Lambda' \to \Lambda'$ by setting $\mathrm{tw}_j(\sigma) = \chi_{\mathrm{cyc}}(\sigma)^j \sigma$ for $\sigma \in \mathrm{Gal}(L_{p,\infty}/K)$ where $\chi_{\mathrm{cyc}} : \mathrm{Gal}(L_{p,\infty}/K) \to \mathbb{Z}_p^\times$ is the cyclotomic character. We denote by $\Lambda' \otimes_j -$ the tensor product over Λ' in which Λ' acts on the first factor via tw_j. This induces a canonical isomorphism (see, for example, [36, Prop. 1.6.5(3)]) between the complexes $\Lambda' \otimes_j C_{L_{p,\infty},S,T}(0)$

and $C_{L_{p,\infty},S,T}(j)$ in $D^{\mathrm{p}}(\Lambda')$. By taking determinants, there is an isomorphism of Λ'-modules $\Lambda' \otimes_j \mathrm{Det}_{\Lambda'}(C_{L_{p,\infty},S,T}(0)) \xrightarrow{\sim} \mathrm{Det}_{\Lambda'}(C_{L_{p,\infty},S,T}(j))$. On the other hand, we denote the natural lift of the idempotent $\varepsilon^+ \in \mathbb{Z}_p[L_p/K]$ in Λ' by ε^+ as well. Now we define the following composition of maps

$$
\begin{aligned}
\mathrm{Tw}_{0,j} : \mathrm{Det}_{\Lambda}(C_{L^+_{p,\infty},S,T}(0)) &= \varepsilon^+ \mathrm{Det}_{\Lambda'}(C_{L_{p,\infty},S,T}(0)) \\
&\to \varepsilon^+(\Lambda' \otimes_j \mathrm{Det}_{\Lambda}(C_{L_{p,\infty},S,T}(0))) \xrightarrow{\sim} \varepsilon^+ \mathrm{Det}_{\Lambda'}(C_{L_{p,\infty},S,T}(j)) = \mathrm{Det}_{\Lambda}(C_{L^+_{p,\infty},S,T}(j))
\end{aligned} \tag{6.5}
$$

where the first arrow is defined by $\varepsilon^+ x \mapsto \varepsilon^+(1 \otimes x)$. The main theorem of this chapter is the following.

Theorem 6.2.1. Let j be a non-zero even integer and χ in $\widehat{\Delta}$. If both

(i) the higher rank Iwasawa main conjecture (Conjecture 6.1.2) is valid for $L^+_{p,\infty}/K$, and

(ii) the Generalised Kummer Congruences are valid for the data $(L_\chi(\mu_{p^n})/K,\ \varepsilon^+,\ 0,\ j)$ for every n,

then the image under $\mathrm{Tw}_{0,j}$ of the element $\mathcal{L}_0$ in (i) is a Λ-basis of $\mathrm{Det}_{\Lambda}(C_{L^+_{p,\infty},S,T}(j))$ with the following property: for every ψ in $\widehat{\mathcal{G}_\chi}$, one has $\lambda^{\psi}_{j}(\mathrm{Tw}_{0,j}(\mathcal{L}_0)) = L^*_{K,S,T}(\psi^{-1}, j)$.

Remark 6.2.2. The method of proving Theorem 6.2.1 suggests that by assuming the validity of Generalised Kummer Congruences for other integral idempotents instead of ε^+, one can deduce that $\mathcal{L}_0$ interpolates $L^*_{K,S,T}(\psi^{-1}, j)$, but for a much more restricted class of characters ψ and integers j such that λ^{j}_{ψ} is defined (one can extend these classes by assuming the widely open conjectures of Leopoldt and Schneider). In this thesis, we focus on the applications of the Generalised Kummer Congruences with respect to ε^+ because the interpolation property will neither depend on the conjectures of Leopoldt or Schneider (which are difficult to verify) and very interesting cases of these congruences are known to be true unconditionally.

6.3 Proof of Theorem 6.2.1

6.3.1 Twisting at finite levels

In this section, we give an explicit description of the twisting morphism (4.10) that occurs in the statement of Generalised Kummer Congruences by choosing a convenient representative of the complex $C_{M,S,T}(0)$.

Let M/K be a finite abelian extension and write $G = \mathrm{Gal}(M/K)$. Fix a positive integer n such that $\mu_{p^n} \subset M^\times$ and write $\mathrm{tw}_j^{M/K}$ to be the ring automorphism of $\mathbb{Z}/p^n[G]$ defined by $\sigma \mapsto \chi_{cyc}^{M/K}(\sigma)^j\sigma$.

By a standard argument (see, for example, [16, §5.4] or [8, Lem. 6.1]), the complex $C_{M,S,T}(0)$ in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ has a representative of the form $F \xrightarrow{\psi_0} F$ concentrated at degree 1 and 2, where F is a free $\mathbb{Z}_p[G]$ of sufficiently large rank d. Let $\{b_1, b_2, ..., b_d\}$ be a $\mathbb{Z}_p[G]$-basis of F. Define the $\mathrm{tw}_j^{M/K}$-semilinear map $\mathrm{Tw}_j^{M/K} : F/p^n \to F/p^n$ given explicitly by

$$\sum_{i=1}^{d} \alpha_i b_i \mapsto \sum_{i=1}^{d} \mathrm{tw}_j^{M/K}(\alpha_i) b_i. \tag{6.6}$$

One can regard $H^1_T(\mathcal{O}_{M,S}, \mathbb{Z}_p(1)) = H^1(C_{M,S,T}(0))$ as a $\mathbb{Z}_p[G]$- submodule of F via the natural inclusion

$$\iota : H^1_T(\mathcal{O}_{M,S}, \mathbb{Z}_p(1)) = \ker(\psi_0) \hookrightarrow F. \tag{6.7}$$

For any place $w \in S_\infty(M)$, we write $i_w : M \hookrightarrow \mathbb{C}$ for the associated embedding and denote

$\xi_w := i_w^{-1}(e^{2\pi\sqrt{-1}/p^n}) \in \mu_{p^n}(M)$. For any integer j, define the map

$$\begin{aligned} \iota_{w,j} : H^1_T(\mathcal{O}_{M,S}, \mathbb{Z}/p^n(1-j)) &\xrightarrow{\sim} H^1_T(\mathcal{O}_{M,S}, \mathbb{Z}/p^n(1)) \otimes_{\mathbb{Z}/p^n} \mu_{p^n}^{\otimes -j} \\ &\to F/p^n \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p^n(-j) \\ &\to F/p^n \end{aligned} \tag{6.8}$$

where the first arrow is the isomorphism induced by the cup product morphism with $\xi_w^{\otimes -j}$, the second arrow is induced by the inclusion ι in (6.7) and the identification $\mu_{p^n}^{\otimes -j} \to \mathbb{Z}/p^n(-j)$ given by $\xi_w \mapsto 1$, and the third arrow is defined by setting $(b_i \otimes 1) \mapsto b_i$ for all i. We remark that $\iota_{w,0} = \mathrm{red}_{p^n} \circ \iota$, where red_{p^n} denotes the reduction modulo p^n morphism of a $\mathbb{Z}_p[G]$-module. Since $\iota_{w,0}$ is independent of the place $w \in S_\infty(M)$, in the sequel we will denote it by ι_0.

Lemma 6.3.1. Define $\mathcal{C}_{w,j} : H^1_T(\mathcal{O}_{M,S}, \mathbb{Z}/p^n(1)) \to H^1_T(\mathcal{O}_{M,S}, \mathbb{Z}/p^n(1-j))$ to be the morphism given by $c \mapsto c \cup \xi_w^{\otimes -j}$. Then the following diagram commutes.

$$\begin{tikzcd} H^1_T(\mathcal{O}_{M,S}, \mathbb{Z}/p^n(1)) \arrow[r, "\mathcal{C}_{w,j}"] \arrow[d, "\iota_0"'] & H^1_T(\mathcal{O}_{M,S}, \mathbb{Z}/p^n(1-j)) \arrow[d, "\iota_{w,j}"] \\ F/p^n \arrow[r, "\mathrm{Tw}_j^{M/K}"'] & F/p^n \end{tikzcd}$$

where $\mathrm{Tw}_j^{M/K}$ is as defined in (6.6). Note that the horizontal maps are $\mathrm{tw}_j^{M/K}$-semilinear.

Proof. Let $c \in H^1_T(\mathcal{O}_{M,S}, \mathbb{Z}/p^n(1))$. We write

$$\iota_0(c) = \sum_{i=1}^{d} \alpha_i b_i \text{ for some } \alpha_i = \sum_{\sigma \in G} \alpha_{i,\sigma}\sigma \in \mathbb{Z}/p^n[G].$$

Inside the module $F/p^n \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p^n(-j)$, for any $1 \le i \le d$, one has

$$\begin{aligned} \alpha_i b_i \otimes 1 &= \sum_{\sigma\in G} \alpha_{i,\sigma} \cdot (\sigma b_i \otimes 1) \\ &= \sum_{\sigma\in G} \alpha_{i,\sigma}\sigma \cdot (b_i \otimes \chi_{cyc}^{M/K}(\sigma)^j) \\ &= \sum_{\sigma\in G} \alpha_{i,\sigma} \cdot \chi_{cyc}^{M/K}(\sigma)^j \sigma \cdot (b_i \otimes 1) \\ &= \mathrm{tw}_j^{M/K}(\alpha_i)(b_i \otimes 1). \end{aligned}$$

Using this and the definition of $\iota_{w,j}$ in (6.8), we deduce that

$$\iota_{w,j}(\mathcal{C}_{w,j}(c)) = \sum_{i=1}^{d} \mathrm{tw}_j^{M/K}(\alpha_i) b_i.$$

On the other hand, according to the definition of $\mathrm{Tw}_j^{M/K}$ in (6.6), we have

$$\mathrm{Tw}_j^{M/K}(\iota_0(c)) = \mathrm{Tw}_j^{M/K}(\sum_{i=1}^{d} \alpha_i b_i) = \sum_{i=1}^{d} \mathrm{tw}_j^{M/K}(\alpha_i) b_i.$$

This finishes the proof. $\square$

Corollary 6.3.2. Let $c_{w,j} : H_T^1(\mathcal{O}_{M,S}, \mathbb{Z}/p^n(1-j))^* \to H_T^1(\mathcal{O}_{M,S}, \mathbb{Z}/p^n(1))^*$ be the $\mathbb{Z}/p^n[G]$-dual of $\mathcal{C}_{w,j}$. For any $r \ge 0$ and any set of places $\{w_1, w_2, ..., w_r\} \subseteq S_\infty(M)$, the following diagram commutes.

$$\begin{tikzcd} \bigcap^r_{\mathbb{Z}/p^n[G]} H_T^1(\mathcal{O}_{M,S}, \mathbb{Z}/p^n(1)) \arrow[r, "(\bigwedge_{u=1}^r c_{w_u,j})^*"] \arrow[d, "(\bigwedge_{u=1}^r \iota_0^*)^*"'] & \bigcap^r_{\mathbb{Z}/p^n[G]} H_T^1(\mathcal{O}_{M,S}, \mathbb{Z}/p^n(1-j)) \arrow[d, "(\bigwedge_{u=1}^r \iota_{w_u,j}^*)^*"] \\ \bigwedge^r_{\mathbb{Z}/p^n[G]}(F/p^n) \arrow[r, "\bigwedge_{u=1}^r \mathrm{Tw}_j^{M/K}"'] & \bigwedge^r_{\mathbb{Z}/p^n[G]}(F/p^n) \end{tikzcd}$$

Proof. This follows from dualising, taking the r-th exterior power and then dualising again the commutative diagram in Lemma 6.3.1 and the natural identification $\bigwedge^r_{\mathbb{Z}/p^n[G]} F/p^n \xrightarrow{\sim} \left(\bigwedge^r_{\mathbb{Z}/p^n[G]}(F/p^n)^*\right)^*$ given by $m \mapsto (f \mapsto f(m))$. $\square$

6.3.2 Twisting at infinite level

In this section, we are going to take the limit of the results in §6.3.1 over the cyclotomic tower $L^+_{p,\infty}/K$. To do this, we first describe a useful resolution of the complex $C_{L_{p,\infty},S,T}(j)$ for each even integer j.

Lemma 6.3.3. Let j be any even integer. There exists a free Λ'-module F_∞ of rank d with a basis $\{b_1, b_2, ..., b_d\}$ and a Λ'-morphism $\Psi_0 : F_\infty \to F_\infty$ such that a representative of $C_{L_{p,\infty},S,T}(0)$ is given by $F_\infty \xrightarrow{\Psi_0} F_\infty$ (placing at degree 0 and 1). Moreover, if $\Psi_j : F_\infty \to F_\infty$ is defined by setting

$$b^*_m \circ \Psi_j(b_n) = \mathrm{tw}_j(b^*_m \circ \Psi_0(b_n)) \tag{6.9}$$

for all $1 \leq m, n \leq d$, then the complex defined by $F_\infty \xrightarrow{\Psi_j} F_\infty$, concentrated at degree 0 and 1, is isomorphic to $C_{L_{p,\infty},S,T}(j)$ in $D^{\mathrm{p}}(\Lambda')$.

Proof. The first part of the claim is true for some sufficiently large d by a standard argument in homological algebra (see, for example, [16, §5.4]). In the rest of the proof, we fix a representative of $C_{L_{p,\infty},S,T}(0)$ of the form $F_\infty \xrightarrow{\Psi_0} F_\infty$ (placing at degree 0 and 1) where F_∞ is a free Λ'-module of sufficiently large rank d.

The ring automorphism $\mathrm{tw}_j : \Lambda' \to \Lambda'$ induces an extension of scalar of Λ'-modules, which we will denote by $\Lambda' \otimes_j -$ as before. Now it suffices to show that the diagram

$$\begin{tikzcd}
\Lambda' \otimes_j F_\infty \arrow[r, "1 \otimes \Psi_0"] \arrow[d] & \Lambda' \otimes_j F_\infty \arrow[d] \\
F_\infty \arrow[r, "\Psi_j"'] & F_\infty
\end{tikzcd}$$

commutes where the vertical maps are given by assigning $1 \otimes b_m \mapsto b_m$ for each $1 \leq m \leq d$. To prove this, we note that for any $1 \leq n \leq d$,

$$\begin{aligned}
1 \otimes \Psi_0(b_n) &= 1 \otimes \sum_{m=1}^{d} (b_m^* \circ \Psi_0(b_n))b_m \\
&= \sum_{m=1}^{d} \left(1 \otimes (b_m^* \circ \Psi_0(b_n))\, b_m\right) \\
&= \sum_{m=1}^{d} (b_m^* \circ \Psi_0(b_n))(1 \otimes b_m) \\
&= \sum_{m=1}^{d} (\mathrm{tw}_j(b_m^* \circ \Psi_0(b_n)) \otimes b_m) \\
&= \sum_{m=1}^{d} \mathrm{tw}_j(b_m^* \circ \Psi_0(b_n))(1 \otimes b_m)
\end{aligned}$$

On the other hand, by the definition of Ψ_j in (6.9), one has

$$\Psi_j(b_n) = \sum_{m=1}^{d} \mathrm{tw}_j(b_m^* \circ \Psi_0(b_n))b_m,$$

for each $1 \leq n \leq d$. So we conclude that the desired diagram commutes. $\square$

Now we define for each even integer j a canonical projection morphism that extends the one defined in [17, §2.4]. To do this, we fix $\chi \in \widehat{\Delta}$ and an even integer j. Then for each

n, we abbreviate $L_n := L_\chi(\mu_{p^n})$, $G_n := \mathrm{Gal}(L_n/K)$ and $G_n^+ := \mathrm{Gal}(L_n^+/K)$. Recall that $r = [K:\mathbb{Q}]$. We then define for each n a canonical projection (here we abbreviate $\mathrm{Det}_{\mathbb{Z}_p[G_n]}$ (resp. $\mathrm{Det}_{\mathbb{Z}_p[G_n^+]}$) to D_n (resp. D_n^+) and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} -$ to $\mathbb{Q}_p \cdot -$) by setting

$$\begin{aligned}
\pi^j_{\chi,n} : \mathrm{Det}_\Lambda(C_{L^+_{p,\infty},S,T}(j)) \to\ & D_n^+(C_{L_n^+,S,T}(j)[1]) \\
& \xrightarrow{\mathbb{Q}_p\varepsilon_j^+\cdot} \varepsilon_j^+\mathbb{Q}_p D_n^+(C_{L_n^+,S,T}(j)[1]) \\
& \xrightarrow{\sim} \varepsilon_j^+\varepsilon^+\mathbb{Q}_p D_n(C_{L_n,S,T}(j)[1]) \\
& \xrightarrow{\sim} \varepsilon_j^+(\mathbb{Q}_p D_n(H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}_p(1-j))) \otimes_{\mathbb{Q}_p[G_n]} \mathbb{Q}_p D_n(Y_{L_n}(j))) \\
& \xrightarrow{\sim} \varepsilon_j^+\mathbb{Q}_p \bigwedge^r_{\mathbb{Z}_p[G_n]} H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}_p(1-j)) \\
& \subseteq \varepsilon^+\mathbb{Q}_p \bigwedge^r_{\mathbb{Z}_p[G_n]} H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}_p(1-j)).
\end{aligned}$$

Here the first arrow is induced by the natural descent morphism, the second arrow is the base change morphism to $\mathbb{Q}_p\varepsilon_j^+$ (see (4.9) for the definition of ε_j^+), the third arrow is the base change property of the determinant functor, the fourth one is the 'passage to cohomology' property of the determinant functor and the equality $\varepsilon_j^+\varepsilon^+ = \varepsilon_j^+$, the fifth map is the (non-canonical) isomorphism

$$x \otimes w_1 \wedge w_2 \wedge ... \wedge w_r \mapsto x$$

where $\{w_1, ..., w_r\}$ is the fixed basis of $\varepsilon_j^+ Y_{L_n}(j)$ described in §4.2.2.

Remark 6.3.4. Note that when $j = 0$, the map $\pi^0_{\chi,n}$ coincides with $\pi^{S_\infty}_{L_{\chi,n}/K,S,T}$ defined in [17, §2.4] precomposed with an appropriate descent morphism.

Upon taking coinvariants and multiplying by ε^+ of the representative of $C_{L_{p,\infty},S,T}(j)$ in Lemma 6.3.3, there exists a free $\mathbb{Z}_p[G_n]\varepsilon^+$-module F with a basis $\{b_i\}_{1\le i\le d}$ such that $F \xrightarrow{\psi_j} F$

(placed at degree 0 and 1) is a representative of $\varepsilon^+ C_{L_n,S,T}(j)[1]$. Here ψ_j is the induced map of Ψ_j defined in Lemma 6.3.3.

Proposition 6.3.5. If we regard $\varepsilon^+ H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}_p(1-j))$ as a submodule of F via (6.7), $\pi^j_{\chi,n}$ induces a morphism between

$$\mathrm{Det}_\Lambda(C_{L^+_{p,\infty},S,T}(j)) \to \varepsilon^+ \bigcap_{\mathbb{Z}_p[G_n]}^{r} H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}_p(1-j))$$

and this morphism can be explicitly given by

$$\begin{aligned} & b_1 \wedge \cdots \wedge b_d \otimes b_1^* \wedge \cdots \wedge b_d^* \\ & \qquad \mapsto (-1)^{r(d-r)} \sum_{\sigma \in \mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma) \det(b_h^* \circ \psi_j(b_{\sigma(k)}))_{r<h,k\le d} b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(r)} \end{aligned}$$

where $\mathfrak{S}_{d,r} := \{\sigma \in S_d : \sigma(1) < \cdots < \sigma(r) \text{ and } \sigma(r+1) < \cdots < \sigma(d)\}$.

Proof. The first and second assertions follow from the more general results proven in [17, Lem. 2.6(iii)] and [16, Lem. 4.3] respectively. □

Proposition 6.3.6. Let j be an even integer. For any $\chi \in \widehat{\Delta}$ and positive integer n, the following diagram commutes.

$$\begin{tikzcd} \mathrm{Det}_\Lambda(C_{L^+_{p,\infty},S,T}(0)) \arrow[r, "\mathrm{Tw}_{0,j}"] \arrow[d, "\pi^0_{\chi,n}"'] & \mathrm{Det}_\Lambda(C_{L^+_{p,\infty},S,T}(j)) \arrow[d, "\mathrm{red}_{p^n} \circ \pi^j_{\chi,n}"] \\ \varepsilon^+ \bigcap\limits_{\mathbb{Z}_p[G_n]}^{r} H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}_p(1)) \arrow[r, "\varepsilon^+(\mathrm{tw}^{L_n/K}_{0,j})"] & \varepsilon^+ \bigcap\limits_{\mathbb{Z}/p^n[G_n]}^{r} H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}/p^n(1-j)) \end{tikzcd}$$

(see §4.4 for the definition of $\mathrm{tw}^{L_n/K}_{0,j}$)

Proof. We fix a representative $C_{L_{p,\infty},S,T}(0)$ and $C_{L_{p,\infty},S,T}(j)$ of the form $F_\infty \xrightarrow{\Psi_0} F_\infty$ and $F_\infty \xrightarrow{\Psi_j} F_\infty$ respectively as in Lemma 6.3.3. Upon taking coinvariants, the representatives of $C_{L_n,S,T}(0)[1]$ and $C_{L_n,S,T}(j)[1]$ have the form $F \xrightarrow{\psi_0} F$ and $F \xrightarrow{\psi_j} F$ respectively (placed at degree 0 and 1) where ψ_0 and ψ_j are induced map from Ψ_0 and Ψ_j, F is a free $\mathbb{Z}_p[G_n]$-module of rank d.

We set $W := \{w_1, ..., w_r\}$ to be the fixed basis of $\varepsilon_j^+ Y_{L_n}(j)$ described in §4.2.2. By Corollary 6.3.2, the following diagram commutes.

$$\begin{tikzcd}
\bigcap\limits_{\mathbb{Z}_p[G_n]}^{r} H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}_p(1)) \arrow[r, "\mathrm{tw}_{0,j}^{L_n/K}"] \arrow[d, "(\bigwedge^r \iota_0^*)^*"'] & \bigcap\limits_{\mathbb{Z}/p^n[G_n]}^{r} H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}/p^n(1-j)) \arrow[d, "\iota_{W,j}"] \\
\bigwedge\limits_{\mathbb{Z}_p[G_n]}^{r} F \arrow[r, "\bigwedge^r \mathrm{Tw}_j^{L_n/K} \circ \mathrm{red}_{p^n}"'] & \bigwedge\limits_{\mathbb{Z}/p^n[G_n]}^{r} (F/p^n)
\end{tikzcd}$$

(Here $\iota_{W,j}$ is the $\mathbb{Z}/p^n[G]$-dual of the map $\bigwedge^r_{\mathbb{Z}/p^n[G]}(F/p^n)^* \to \bigwedge^r_{\mathbb{Z}/p^n[G]}(H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}/p^n(1-j))^*$ that sends each element $\wedge_{u=1}^{u=r} x_u$ to $\wedge_{u=1}^{u=r} \iota_{w_u,j}^*(x_u)$.) Hence, it suffices to show that the diagram

$$\begin{tikzcd}
\mathrm{Det}_\Lambda(C_{L^+_{p,\infty},S,T}(0)) \arrow[r, "\mathrm{Tw}_{0,j}"] \arrow[d, "\widetilde{\pi^0_{\chi,n}}"'] & \mathrm{Det}_\Lambda(C_{L^+_{p,\infty},S,T}(j)) \arrow[d, "\widetilde{\pi^j_{\chi,n}}"] \\
\varepsilon^+ \bigwedge\limits_{\mathbb{Z}_p[G_n]}^{r} F \arrow[r, "\varepsilon^+(\bigwedge^r \mathrm{Tw}_j^{L_n/K} \circ \mathrm{red}_{p^n})"'] & \varepsilon^+ \bigwedge\limits_{\mathbb{Z}/p^n[G_n]}^{r} (F/p^n)
\end{tikzcd}$$

where $\widetilde{\pi^0_{\chi,n}} = (\varepsilon^+(\bigwedge^r \iota_0^*)^*) \circ \pi^0_{\chi,n}$ and $\widetilde{\pi^j_{\chi,n}} = (\varepsilon^+(\iota_{W,j})) \circ \mathrm{red}_{p^n} \circ \pi^j_{\chi,n}$, is commutative.

To prove this, we first let $x = b_1 \wedge \cdots \wedge b_d \otimes b_1^* \wedge \cdots \wedge b_d^*$. Then we have

$$\begin{aligned}
\widetilde{\pi^j_{\chi,n}} \circ \mathrm{Tw}_{0,j}(x) &= \varepsilon^+((-1)^{r(d-r)} \sum_{\sigma \in \mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma) \det(b_h^* \circ \psi_j(b_{\sigma(k)})))_{r<h,k\le d} b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(r)}) \\
&= \varepsilon^+((-1)^{r(d-r)} \sum_{\sigma \in \mathcal{S}_{d,r}} \mathrm{sgn}(\sigma) \det(\mathrm{tw}_j^{L_n/K}(b_h^* \circ \psi_0(b_{\sigma(k)})))_{r<h,k\le d} b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(r)})
\end{aligned}$$

where the first equality follows from Proposition 6.3.5 and the second equality follows from Lemma 6.3.3. On the other hand, using Proposition 6.3.5 again, we have

$$\widetilde{\pi^0_{\chi,n}(x)} = \varepsilon^+((-1)^{r(d-r)} \sum_{\sigma \in \mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma) \det(b_h^* \circ \psi_0(b_{\sigma(k)}))_{r<h,k\le d} b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(r)}).$$

Now by the definition of $\mathrm{Tw}_j^{L_n/K}$ in (6.6), one has

$$\begin{aligned}
&\bigwedge^r \mathrm{Tw}_j^{L_n/K} \circ \mathrm{red}_{p^n} \circ \widetilde{\pi^0_{\chi,n}(x)} \\
&= \varepsilon^+((-1)^{r(d-r)} \sum_{\sigma \in \mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma) \det(\mathrm{tw}_j^{L_n/K}(b_h^* \circ \psi_0(b_{\sigma(k)})))_{r<h,k\le d} b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(r)}).
\end{aligned}$$

This finishes the proof. $\square$

We are now ready to deduce the central commutative diagram as follows.

Corollary 6.3.7. Let $\chi \in \Delta$. The following diagram commutes.

$$\begin{tikzcd}
\mathrm{Det}_\Lambda(C_{L^+_{p,\infty},S,T}(0)) \arrow[r, "\mathrm{Tw}_{0,j}"] \arrow[d, "\pi^0_{\chi,\infty}"'] & \mathrm{Det}_\Lambda(C_{L^+_{p,\infty},S,T}(j)) \arrow[d, "\pi^j_{\chi,\infty}"] \\
\varprojlim_n \varepsilon^+ \bigcap^r_{\mathbb{Z}_p[G_n]} H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}_p(1)) \arrow[r, "\varprojlim_n \mathrm{tw}_{0,j}^{L_n/K}"'] & \varprojlim_n \varepsilon^+ \bigcap^r_{\mathbb{Z}/p^n[G_n]} H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}/p^n(1-j))
\end{tikzcd}$$

where the vertical maps are given by $\pi^a_{\chi,\infty} := \varprojlim_n \pi^a_{\chi,n}$ (the inverse limit is taken with respect to the field theoretical norm maps).

Proof. This is deduced by taking the inverse limit of the commutative diagram in Proposition 6.3.6 over all the immediate finite sub-extensions L_n/K of the cyclotomic tower $L_{\chi,\infty}/K$ where $L_{\chi,\infty} := L^+_\chi K_\infty$. $\square$

6.3.3 Completion of the proof

We are now ready to finish the proof of Theorem 6.2.1 by using the commutative diagram in Corollary 6.3.7. We adopt the notation from §4.3. In particular, we have fixed a character $\chi \in \widehat{\Delta}$. Fix any non-zero even integer j and set $\mathcal{L}_j = \mathrm{Tw}_{0,j}(\mathcal{L}_0)$.

By the assumed validity of the higher rank main conjecture, there exists a Λ-basis $\mathcal{L}_0$ of $\mathrm{Det}_\Lambda(C_{L^+_{p,\infty},S,T}(0))$ with the prescribed interpolation properties. In particular, this directly implies that (see [17, Th. 3.4]) $\pi^0_{\chi,\infty}(\mathcal{L}_0) = \varprojlim_n \eta^{S_\infty}_{L^+_n/K,S,T}$ where $\eta^{S_\infty}_{L^+_n/K,S,T}$ is the Rubin-Stark element for the data $(L^+_n/K, S, T, S_\infty)$, after completion at p. According to Remark 4.2.7, this element coincides with the Stark element $\eta^{\varepsilon^+}_{L_n/K,S,T}(0)$ and hence we have the equality

$$\pi^0_{\chi,\infty}(\mathcal{L}_0) = \varprojlim_n \eta^{\varepsilon^+}_{L_n/K,S,T}(0). \tag{6.10}$$

On the other hand, the validity of the Generalised Kummer Congruences for the data $(L_\chi(\mu_{p^n})/K,\ \varepsilon^+,\ 0,\ j)$ for all n implies that there is an equality

$$\varprojlim_n \mathrm{tw}_{0,j}(\eta^{\varepsilon^+}_{L_n/K,S,T}(0)) = \varprojlim_n \eta^{\varepsilon^+}_{L_n/K,S,T}(j). \tag{6.11}$$

According to Lemma 2.6.3, the above equality can be regarded as an equality in the module $\varprojlim_n \varepsilon^+ \bigcap^r_{\mathbb{Z}_p[G_n]} H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}_p(1-j))$. The commutative diagram in Corollary 6.3.7 and the equalities (6.10) and (6.11) thus imply that

$$\pi^j_{\chi,\infty}(\mathcal{L}_j) = \varprojlim_n \eta^{\varepsilon^+}_{L_n/K,S,T}(j). \tag{6.12}$$

Now we take any $\psi \in \widehat{\mathcal{G}_\chi}$ and choose n large enough such that L_ψ is contained in L_n^+. We observe that the trivialisation homomorphism λ^ψ_j coincides with the following composition (noting that we assume $j \neq 0$, one has $\varepsilon^+_j = \varepsilon^+$ by Remark 4.2.5)

$$\begin{aligned}
\mathrm{Det}_\Lambda(C_{L^+_{p,\infty},S,T}(j)) &\to \varepsilon^+ \,\mathrm{Det}_{\mathbb{C}_p[G_n]}(\mathbb{C}_p H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}_p(1-j)) \\
&\xrightarrow{\sim} \varepsilon^+(\mathrm{Det}_{\mathbb{C}_p[G_n]}(\mathbb{C}_p H^1_T(\mathcal{O}_{L_n,S}, \mathbb{Z}_p(1-j)) \otimes_{\mathbb{C}_p[G_n]} \mathrm{Det}^{-1}_{\mathbb{C}_p[G_n]}(\mathbb{C}_p Y_{L_n}(-j))) \\
&\xrightarrow{\varepsilon^+(\lambda_j \otimes \mathrm{id})} \varepsilon^+(\mathrm{Det}_{\mathbb{C}_p[G_n]}(\mathbb{C}_p Y_{L_n}(-j)) \otimes_{\mathbb{C}_p[G_n]} \mathrm{Det}^{-1}_{\mathbb{C}_p[G_n]}(\mathbb{C}_p Y_{L_n}(-j))) \\
&\xrightarrow{\sim} \varepsilon^+ \mathbb{C}_p[G_n] \\
&\to \mathbb{C}_p[G_\psi] \\
&\xrightarrow{\psi} \mathbb{C}_p
\end{aligned}$$

where the first arrow is induced by $\mathbb{C}_p \otimes \pi^j_{\chi,n}$, the second arrow is given by

$$x \mapsto x \otimes (w_1^* \wedge w_2^* \wedge ... \wedge w_r^*)$$

in which $\{w_1, w_2, ..., w_r\}$ is the basis for $\varepsilon^+ Y_{L_n}(-j)$ that we fixed previously in §4.2.2, λ_j in the third arrow is the period regulator isomorphism at the value j (see §4.2.2 for its definition), the fourth arrow is the evaluation map, the fifth arrow is induced by the natural projection $\mathbb{C}_p[G_n] \to \mathbb{C}_p[G_\psi]$ (Here the image of ε^+ is 1 because L_ψ is totally real).

By (6.12), the image of $\mathcal{L}_j$ under the composition of the first three arrows is given by

$$\varepsilon^+ \cdot \theta^*_{L_n/K,S,T}(j) \cdot (\wedge^{i=r}_{i=1} w_i) \otimes (\wedge^{i=r}_{i=1} w_i^*)$$

Furthermore, by looking at the image of the above element under the last three arrows, we obtain

$$\lambda_{\psi}^{j}(\mathcal{L}_j) = \psi(\theta^{*}_{L_\psi/K,S,T}(j)) = L^{*}_{K,S,T}(\psi^{-1}, j).$$

This finishes the proof of Theorem 6.2.1.

6.4 A rank one example

In this section, we specialise the main theorem (Theorem 6.2.1) in the case when $K = \mathbb{Q}$. We adopt the notation from the beginning of §6.2.

Theorem 6.4.1. Let p be an odd prime, $L = \mathbb{Q}(\mu_p)^+$ and $S = \{\infty\} \cup \{p\}$. There exists a Λ-basis $\mathcal{L}_0$ of $\mathrm{Det}_\Lambda(C_{L_\infty,S}(0))$ such that for any even integers j, $\chi \in \widehat{\Delta}$ and $\psi \in \widehat{\mathcal{G}_\chi}$,

$$\lambda_j^{\psi}(\mathrm{Tw}_{0,j}(\mathcal{L}_0)) = \delta(\psi, j) \cdot L^{*}_{\mathbb{Q},S}(\psi^{-1}, j)$$

where

$$\delta(\psi, j) = \begin{cases} 0 & \text{if } j = 0 \text{ and } r_{\psi,S}(0) > 1 \\ 1 & \text{otherwise} \end{cases}.$$

Proof. Since p is odd, the module $\varepsilon^+ H^1(\mathcal{O}_{\mathbb{Q}(\mu_{p^n}),S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-torsion-free for every even integer j. Therefore, we may assume T to be empty. According to Theorem 6.1.4, the higher rank Iwasawa Main Conjecture for $L_\infty/\mathbb{Q}$ can be deduced from the main result of Mazur and Wiles in [53]. By Theorem 4.4.3 (i), for any even integers j and $\chi \in \widehat{G}$, the Generalised Kummer Congruence is known for the data $(L_\chi(\mu_{p^n})/\mathbb{Q},\ \varepsilon^+,\ 0,\ j)$ for all n. Therefore, Theorem 6.2.1 implies that $\mathcal{L}_0$ has the desired interpolation properties whenever j is a non-zero even integer. The interpolation property at $j = 0$ follows directly from the statement of the higher rank Iwasawa Main Conjecture. $\square$

Remark 6.4.2. In the view of Theorem 6.4.1 (and its proof), one can regard the interpolation formula that appears in Theorem 6.2.1 as a refinement of the higher rank Iwasawa Main Conjecture of Burns, Kurihara and Sano in [17]. In particular, it is natural to expect the Iwasawa theoretical zeta element has interpolation properties at various weights. Such a philosophy is coherent with the very general (and much stronger) conjecture formulated by Fukaya and Kato in [36].

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